Sequential Global Games

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Abstract

Global games are incomplete information games where players receive private noisy signals about the true game played. In a sequential global game, the set of players is partitioned into subsets. Players within a subset (of the partition) play simultaneously but no two subsets move at the same time. The resulting sequence of stages introduces intricate dynamics not encountered in simultaneous move global games. We show that a sequential global game with strategic complementarities and binary actions has at least one equilibrium in increasing strategies. A sequential global game always has an equilibrium in increasing strategies. However, even with vanishing noise the general equilibrium uniqueness of one-shot global games breaks down: even a simple two-stage sequential global game does not generally have a unique equilibrium surviving iterated dominance. The reason is that the history of play may force players to believe that the true game is in fact "far away" from their signal – a possibility that does not arise in one-shot game. We identify sufficient conditions for equilibrium uniqueness.

1 Introduction

Global games are a class of incomplete information games where players receive private noisy signals about the true game being played. Introduced by Carlsson and Van Damme

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(1993), the approach has been applied to such wide-ranging phenomena as currency attacks (Morris and Shin, 1998), regime switches (Chamley, 1999; Angeletos et al., 2007), financial crises (Angeletos and Werning, 2006), political protests (Edmond, 2013), bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005), and platform markets Jullien and Pavan (2019). This paper introduces sequential global games with strategic complementarities and binary actions.

In a sequential global game, the player set is partitioned into (not necessarily singleton) subsets. Once the true game is drawn and private signals are received, players choose their actions. While players within a subset play simultaneously, no two subsets move at the same time. Instead, different subsets move in a given order of stages and each subset moves only once. In any stage, the history of play in earlier stages is observed. Individual payoffs depend on the true game drawn and the actions chosen by all players.

The timing of sequential global games is different from that studied in simultaneous move (Carlsson and Van Damme, 1993; Frankel et al., 2003) or repeated (Angeletos et al., 2007) global games. In a simultaneous move game, all players move once and simultaneously after receiving their signals. In a repeated game, all players move simultaneously as well but they do so multiple times, and each stage game is drawn separately.

We restrict attention to sequential global games with strategic complementarities (Bulow et al., 1985) and binary actions. Games of regime change are an often-studied special case of games in this class, applications including currency attacks (Morris and Shin, 1998), regime switches (Chamley, 1999; Angeletos et al., 2007), financial crises (Angeletos and Werning, 2006), political protests (Edmond, 2013), and bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005). Strategic complementarities cut to the core of equilibrium multiplicity and coordination failure in complete information coordination games (Cooper and John, 1988; Van Huyck et al., 1990).

While one-shot sequential global games are guaranteed to have a unique equilibrium when the noise vanishes, this strong general result breaks down for sequential global games. Though it is of course possible to construct specific sequential global games with strategic complementarities and binary actions, iterated dominance does not generally reduce the set of equilibria to a unique strategy profile. Besides this negative result, however, we also derive more positive results. First, a sequential global game always has at least one perfect Bayesian equilibrium in increasing strategies. Moreover, if the noise in players' signals vanishes, the game has only one perfect Bayesian equilibrium in increasing strategies that survives iterated dominance. (Note the implication: if , in the limit, a sequential global game has multiple equilibria, only one will be in increasing strategies).

Sequential global games have intricate dynamics not encountered in simultaneous move environments. Consider a two-stage game of staggered investments in some novel network technology, where uncertainty and signals pertain to the technology's usefulness or quality.[1] At the start of stage 2, firms observe the investments made in stage 1. Not only does this observation establish a minimum network size for the technology (stimulating stage 2 investments), it also provides indirect evidence of the technology's quality. In equilibrium, these effects are mutually reinforcing. Firms in stage 1 will adopt the technology only if its quality is perceived to be high. In stage 2, large first-stage investments are then good news for two reasons: (i) there is a sizable network to join by investing, and (ii) the technology's true quality is likely to be high. Firms in stage 2 will therefore invest even for relatively low private quality assessments (the large network and high quality assessment of first-stage firms make up for low private estimates). The readiness of stage 2 firms to invest in turn affects investment in the first stage. This leads to an intricate cycle where investment decisions in different stages influence each other back and forth.

Due to the intricate dynamics of a sequential global game, equilibrium uniqueness is not longer guaranteed (Angeletos et al., 2007). However, contrary to the game of Angeletos et al. (2007), we do not need an infinite number of stages or players. Already with a finite number of players and only two stages, the game can have multiple equilibria. As an antidote to this general negative result, we provide sufficient conditions for the game to have a unique equilibrium.

2 The Game

Let the set of players be $\mathbb{P} = \{1, 2, ..., N\}$. Each agent $i \in \mathbb{P}$ chooses action $x_i \in \{0, 1\}$. We define $\boldsymbol{x} := (x_i)_{i \in \mathbb{P}}$. The vector of actions played by all players but i is:

$$\boldsymbol{x}_{\neg i} := (x_j)_{j \in \mathbb{P} \setminus \{i\}}. \tag{2.1}$$

[1]The Electronic Medical Record would be an example of such a technology, see Dranove et al. (2014).

If player i plays x_i while the other players play $\boldsymbol{x}_{\neg i}$, the payoff to player i is given by $u(x_i, \boldsymbol{x}_{\neg i}, \beta)$, where β is a payoff-relevant parameter. The payoff function u is symmetric with respect to each x_j in $\boldsymbol{x}_{\neg i}$ and the same for all players i.[2] Due to the symmetry of players, the payoff to player i depends only on the total number of players j player $x_j = 1$, i.e. $u(x_i, n, \beta)$ where $n := \sum_{j \neq i} x_j$. Note that $n \in \{0, 1, ..., N-1\}$.

Player i's gain from playing $x_i = 1$ instead of $x_i = 0$, given β and n, is:

$$G(n, \beta) = u(1, n, \beta) - u(0, n, \beta). \tag{2.2}$$

We make the following assumptions:

(*A1*) *The function G is strictly increasing in β, for all n, for all i.*

(*A2*) *The function G is strictly increasing in n, for all β, for all i.*

(*A3*) *There exist points $\beta_0 > -\infty$ and $\beta_1 < \infty$ such that, for all i, $G(N-1, \beta_0) = 0$ and $G(0, \beta_1) = 0$.*

Assumption (A2) implies that players' actions are strategic complements.[3] Strategic complementarities cut to the core of equilibrium multiplicity in coordination games where β is common knowledge (Cooper and John, 1988; Van Huyck et al., 1990). By maintaining assumption (A2), we turn the odds of finding a unique equilibrium against ourselves.

While payoffs depend on β, this parameter is unobserved and drawn from a normal distribution $\mathcal{N}(\bar{\beta}, \sigma_\beta^2)$. Each player i receives a private noisy signal b_i of β, such that:

$$b_i = \beta + \varepsilon_i, \tag{2.3}$$

where ε_i is a noise term drawn i.i.d. from a normal distribution $\mathcal{N}(0, \sigma_\varepsilon^2)$. We write $F^\varepsilon(\beta, \boldsymbol{b}_{\neg i} \mid b_i)$ for the conditional posterior distribution of $(\beta, \boldsymbol{b}_{\neg i})$, given b_i.

Lemma 1. *Given b_i, the vector $(\beta, \boldsymbol{b}_{\neg i}) \sim \mathcal{N}_n((1-\lambda)\bar{\beta} + \lambda b_i), \Sigma')$, where $\lambda := \sigma_\beta^2 / (\sigma_\varepsilon^2 + \sigma_\beta^2)$.*[4] *Importantly,*

[2]These assumptions are restrictive but simplify the analysis substantially. See Morris and Shin (1998, 2002); Angeletos and Pavan (2004); Angeletos and Werning (2006); Angeletos et al. (2007) for examples of other papers that operate under these same assumptions.

[3]For an interesting recent developments of (simultaneous move) global games with strategic substitutes, see Harrison and Jara-Moroni (2020).

[4]The covariance matrix Σ' essentially has two parts. First, a smaller $(n-1) \times (n-1)$ covariance

(i) The mean vector of $(\beta, \boldsymbol{b}_{\neg i})$ shifts linearly with b_i;

(ii) The covariance matrix Σ' is independent of b_i.

Proof. These are standard properties of the (multivariate) normal distribution. See for example Tong (2012).

Note that $\sigma_\varepsilon \to 0$ implies $\lambda \to 1$ and so $(1-\lambda)\bar{\beta} + \lambda b_i \to (b_i, b_i, ..., b_i)$, that is, when signals become arbitrarily precise players expect both the real β as well as the signals received by all others to be Normally distributed about their own.

3 Simultaneous Moves as a Benchmark

To serve as a benchmark for eventual comparison, this section briefly discuss a simultaneous move global game. The analysis and results presented here are a special case of Frankel et al. (2003), with the exception that our noise has support on the entire real line (even in the limit) whereas Frankel, Morris, & Pauzner assume a strictly bounded noise-support. In Section **??**, we transform the simultaneous move game into a sequential game by partitioning the player set into subsets, leaving the remaining structure of the game as is.

The structure of a simultaneous move global game is common knowledge and as follows:

1. Nature draws a true β.
2. Each $i \in \mathbb{P}$ receives private signal $b_i = \beta + \varepsilon_i$ of β.
3. All $i \in \mathbb{P}$ simultaneously play action $x_i \in \{0, 1\}$.
4. Payoffs are realized according to β and the actions chosen by all players.

In the simultaneous move game, all players choose their actions simultaneously. Since this eliminates the possibility that the action profile $\boldsymbol{x}_{\neg i}$ depends on the realized x_i, we write $G(\boldsymbol{x}_{\neg i}, \beta)$ for the conditional gain of player i.

A pure strategy for player $i \in \mathbb{P}$ is a mapping $s_i : \mathbb{R} \to \{0, 1\}$ projecting signals onto actions. We write S_i for the set of strategies for player i, and $S_{\neg i}$ for the set of strategy

matrix Σ'' for the signals whose diagonal elements are $(\sigma^4 - \sigma_\beta^4)/\sigma^2$ and whose off-diagonal elements $(\sigma^2\sigma_\beta^2 - \sigma_\beta^4)/\sigma^2$. Second, a vector $(\sigma_\varepsilon^2 + \sigma_\beta^2, ...)$ for the variance of β and its covariance with all signals b_j for $j \neq i$, which is the first column/row of Σ'.

profiles for all players $j \in \mathbb{P} \setminus \{i\}$, with typical element $\boldsymbol{s}_{\neg i}$. Let $\boldsymbol{b} \in \mathbb{R}^n$ denote a vector $(b_i)_{i\in\mathbb{P}}$ of signals b_i for all agents i, and let $\boldsymbol{b}_{\neg i}$ be a vector of signals for all players but i.

Let player i's expected gain, conditional on b_i and the strategies $s_{\neg i}$ played by all other players, for given σ_ε, be denoted:

$$g^\varepsilon(\boldsymbol{s}_{\neg i}, b_i) := \iint G(\boldsymbol{s}_{\neg i}(\boldsymbol{b}_{\neg i}), \beta)\, dF^\varepsilon(\beta, \boldsymbol{b}_{\neg i} \mid b_i). \tag{3.1}$$

3.1 Monotone Strategies and Equilibrium

For $c \in \mathbb{R}$, let $\tilde{c}$ be the *monotone strategy* defined by $\tilde{c}(b) = 0$ if $b < c$ and $\tilde{c}(b) = 1$ if $b > c$. Everything else constant, the expected gain to playing $x_i = 1$, rather than $x_i = 0$, is increasing in b_i, so it is natural to look at monotone strategies. The monotone strategy profile in which all players play $\tilde{c}$ is denoted $\tilde{\boldsymbol{c}}$.

If all players $j \neq i$ play a monotone strategy, then player i's expected gain is strictly increasing in his signal b_i. Formalized as Lemma 7 in the appendix, this result has a clear economic intuition. If b_i increases, player i's posterior on both β and $\boldsymbol{b}_{\neg i}$ shifts to the right. When the other players plan a monotone strategy profile, this also implies a shift of i's posterior on n to the right. Since i's gain is strictly increasing in both β and n, his expected gain is strictly increasing in b_i.

A monotone strategy profile $\tilde{\boldsymbol{\beta}}^{\boldsymbol{*}}$ is a Bayesian Nash equilibrium of the simultaneous move global game if and only if it is the solution, for each $i \in \mathbb{P}$, to

$$g^\varepsilon(\tilde{\boldsymbol{\beta}}^{\boldsymbol{*}}_{\neg i}, \beta^*) = 0. \tag{3.2}$$

Proposition 1. *The simultaneous move global game always has a Bayesian Nash equilibrium in monotone strategies.*

3.2 Iterated Dominance and Limit Uniqueness

Conditional on the signal b_i, the action $x_i = 1$ is *dominated* if $g^\varepsilon(\boldsymbol{s}_{\neg i}, b_i) < 0$ for all $\boldsymbol{s}_{\neg i}$. Similarly, the action $x_i = 0$ is conditionally dominated if $g^\varepsilon(\boldsymbol{s}_{\neg i}, b_i) > 0$ for all $\boldsymbol{s}_{\neg i}$. Now consider the set of strategies for player i, S_i. Some strategies in the set will be dominated (for example, (A3) implies it is strictly dominated to play $x_i = 0$ for all $b_i > \beta_1$). Stripping the set S_i of all dominated strategies, we obtain a smaller set $S_i^0 \subseteq S_i$, and this is true for each player i. But we know that no player i will ever play a dominated strategy and attention may thus be restricted to the strategies in S_i^0.

Moreover, a given player i knows that any $s_{\neg i} \notin S^0_{\neg i}$ is dominated for at least one player $j \neq i$. Hence, player i should expect his opponents to play a strategy $s_{\neg i}$ from $S^0_{\neg i}$ only. The restriction to action profiles in $S^0_{\neg i}$, however, may lead to additional strategies becoming dominated for player i. Taking those out, player i's set of undominated strategies becomes $S^1_i \subseteq S^0_i$. Again, this is true for all players and player i knows that his opponents will only play strategy profiles belonging to $S^1_{\neg i}$. Yet other strategies for player i may then become dominated, and so on.

The above can be repeated indefinitely and is called iterated elimination of dominated strategies. Do there exist equilibria in monotone strategies that survive iterated dominance?

Proposition 2. *There exist at least Bayesian Nash equilibrium in monotone strategies that survives iterated elimination of dominated strategies.*

The proof of Proposition 2 requires a rather lengthy construction. It is therefore relegated to Appendix A.

Lemma 2. *Let σ_ε be sufficiently small. For any two points $c, d \in \mathbb{R}$, let $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{d}}$ denote the associated monotone strategy profiles. For all $i \in \mathbb{P}$,*

$$g^\varepsilon(\tilde{\boldsymbol{c}}_{\neg i}, c) > g^\varepsilon(\tilde{\boldsymbol{d}}_{\neg i}, d) \iff c > d. \tag{3.3}$$

Proposition 3. *Let σ_ε be sufficiently small. Then the game has a unique equilibrium $\boldsymbol{s}^*$ surviving iterated elimination of strictly dominated strategies. It is the monotone strategy profile $\tilde{\boldsymbol{\beta}}^*$.*

Proof. An immediate implication of Lemma 2 and Proposition 2.

Proposition 3 is a special case of Theorem 1 in Frankel et al. (2003) on simultaneous move global games with strategic complementarities.

4 Two-Stage Sequential Global Games

We now proceed to sequential global games with two stages where players play sequentially according to an ordered partition $\mathscr{P} = \{\{\mathbb{P}_1\}, \{\mathbb{P}_2\}\}$ of the player set $\mathbb{P} = \{1, 2, ..., N\}$, $N \geq 2$. Let $|\mathbb{P}_1| = N_1$ and $|\mathbb{P}_2| = N_2$, so that $N_1 + N_2 = N$. Define $\boldsymbol{x}_1 = (x_i)_{i \in \mathbb{P}_1}$, $\boldsymbol{x}_{1\setminus i} = (x_j)_{j \in \mathbb{P}_1 \setminus i}$, $\boldsymbol{x}_2 = (x_j)_{j \in \mathbb{P}_2}$, $\boldsymbol{x}_{2\setminus j} = (x_i)_{i \in \mathbb{P}_2 \setminus j}$. Moreover, a history

h at the beginning of stage 2 is denoted $h = \boldsymbol{x}_1$. By the symmetry of our game, the payoff-relevant characteristic of any action profile $\boldsymbol{x}_t$ is the number of x_is in $\boldsymbol{x}_t$ that are 1. Therefore, for $t = 1, 2$, define $n_t = \sum_{i \in \mathbb{P}_t} x_i$.

The timing of the game is as follows.

1. Nature draws a true β.
2. Each $i \in \mathbb{P}$ receives private signal $b_i = \beta + \varepsilon_i$ of β.
3. All $i \in \mathbb{P}_1$ simultaneously play action $x_i \in \{0, 1\}$.
4. All $i \in \mathbb{P}_2$ observe the history $h = \boldsymbol{x}_1$.
5. All $i \in \mathbb{P}_2$ simultaneously play action $x_i \in \{0, 1\}$.
6. Payoffs are realized according to β and the actions chosen by all players.

For clarity, we use index i for players in stage 1 and index j for players in stage 2.

4.1 Strategies and Gains

A strategy for player $i \in \mathbb{P}_1$ is a function $s_i : \mathbb{R} \to \{0, 1\}$. Once players in the first stage have chosen their actions, the history h is realized, where $h = (x_i)_{i \in \mathbb{P}_1}$ is a vector in $\mathcal{H} = \{0, 1\}^{N_1}$. We write $\boldsymbol{y}_1$ for the profile $(y_i)_{i \in \mathbb{P}_1}$ and $\boldsymbol{y}_{1 \setminus i}$ for the profile $(y_j)_{j \in \mathbb{P}_1 \setminus \{i\}}$. The set of stage 1 strategy profiles is S_1.

Each player $j \in \mathbb{P}_2$ observes both its private signal b_i and the history h, so a strategy for $j \in \mathbb{P}_2$ is a function $s_j : \mathbb{R} \times \mathcal{H} \to \{0, 1\}$. For $j \in \mathbb{P}_2$, we say that a strategy is monotone if it is monotone for every history h. We write $\boldsymbol{y}_{2 \setminus j}$ for the profile $(y_l)_{l \in \mathbb{P}_2 \setminus \{j\}}$. The set of stage 2 strategy profiles is S_2.

For player $i \in \mathbb{P}_1$, the conditional gain is given by:

$$G_1(\boldsymbol{x}_{1 \setminus i}, \boldsymbol{x}_2, \beta) = u(1, \boldsymbol{x}_{1 \setminus i}, \boldsymbol{x}_2(n_{1 \setminus i} + 1), \beta) - u(0, \boldsymbol{x}_{1 \setminus i}, \boldsymbol{x}_2(n_{1 \setminus i}), \beta). \tag{4.1}$$

For player $j \in \mathbb{P}_2$, define the gain from playing $x_j = 1$, rather than $x_j = 0$, to be:

$$G_2(h, \boldsymbol{x}_{2 \setminus j}, \beta) := u(x_j = 1, h, \boldsymbol{x}_{2 \setminus j}, \beta) - u(x_j = 0, h, \boldsymbol{x}_{2 \setminus j}, \beta). \tag{4.2}$$

Lemma 1 specifies the posterior belief on $(\beta, \boldsymbol{b}_{\neg i})$ of player i in stage 1. Let $F_1^{\varepsilon}(\beta, \boldsymbol{b}_{1 \setminus i}, \boldsymbol{b}_2 \mid b_i)$ denote the associated distribution. Conditional on the signal b_i as well as the strategy-vectors $\boldsymbol{s}_{1\setminus i}$ and $\boldsymbol{s}_2$, the expected gain to player $i \in \mathbb{P}_1$ is:

$$g_1^\varepsilon(\boldsymbol{s}_{1\setminus i}, \boldsymbol{s}_2, b_i) = \iiint G_1(\boldsymbol{s}_{1\setminus i}(\boldsymbol{b}_{1\setminus i}), \boldsymbol{s}_2(\boldsymbol{b}_2, h) \mid \beta, \boldsymbol{b}_{1\setminus i}, \boldsymbol{b}_2)\, dF_1^\varepsilon(\beta, \boldsymbol{b}_{1\setminus i}, \boldsymbol{b}_2 \mid b_i). \quad (4.3)$$

(Note that a strategy for players in the second stage depends on the realized history. We need not explicitly integrate the gain over all possible histories since, given $\boldsymbol{s}_{1\setminus i}$, the distribution of h is implied by the distribution of $\boldsymbol{b}_{1\setminus i}$.)

Denote by $\mathcal{B}(h \mid \boldsymbol{s}_1) = \prod_{i\in\mathbb{P}_1}\{b_i \mid s_i(b_i) = h_i\}$, where $h_i = x_i$, i.e. the element of the history h belonging to player i. For player $j \in \mathbb{P}_2$, let $F_2^\varepsilon(\beta, \boldsymbol{b}_{2\setminus j} \mid b_j, \mathcal{B}_1^h(\boldsymbol{s}_1))$ denote the joint posterior distribution on β and $\boldsymbol{b}_{2\setminus j}$, given b_j and $\mathcal{B}_1^h$. The expected gain to player $j \in \mathbb{P}_2$ is then given by:

$$g_2^\varepsilon(h, \boldsymbol{s}_{2\setminus j}, b_j \mid \boldsymbol{s}_1) = \iint G_2(h, \boldsymbol{s}_{2\setminus j}(\boldsymbol{b}_{2\setminus j}, h) \mid \boldsymbol{b}_{2\setminus j}, \beta)\, dF_2^\varepsilon(\beta, \boldsymbol{b}_{2\setminus i} \mid b_j, \mathcal{B}(h \mid \boldsymbol{s}_1)). \quad (4.4)$$

If players in stage 1 play a symmetric strategy profile, i.e. $s_i = s_{i'}$ for all $i, i' \in \mathbb{P}_1$, then the posterior belief players in stage 2 is invariant with respect to permutations of the history vector h. In this case, it suffices to know n_1, the number ones played in the first stage, and we may write $g_2^\varepsilon(n_1, \boldsymbol{s}_{2\setminus j}, b_j \mid \boldsymbol{s}_1)$.

4.2 Increasing Strategies and Perfect Bayesian Equilibrium

Consider the monotone strategy profile $(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2(n_1))$. We say that $(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2(n_1))$ is an *increasing* strategy profile if $\beta_2(N_1) < \beta_2(N_1 - 1) < ... < \beta_2(0)$.[5] That is, each player in stage 1 plays the monotone strategy $\tilde{\beta}_1$. At the end of that stage, a history is realized and n_1 players have played 1. Given n_1, each player in stage 2 plays the monotone strategy with switching point $\beta_2(n_1)$. The switching point to playing 1 in stage 2 is lower if more players in stage 1 have played 1.

Under an increasing strategy profile, players' posterior expected gains are strictly increasing in their private signals. Moreover, stage 2 players' expected gains are strictly increasing in n_1. The increasing strategy profile $(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2(n_1))$ is hence a perfect Bayesian equilibrium of the two-stage sequential global game if and only if it solves, for each

[5] The strategy is called increasing since the action it prescribes is non-decreasing in both a player's private signal and the number of ones played in stage 1.

$i \in \mathbb{P}_1$ and $j \in \mathbb{P}_2$, the following system of $N_1 + 2$ equations:

$$\begin{aligned} &g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1\backslash i}, \tilde{\boldsymbol{\beta}}_2(), \beta_1) = 0 \\ &g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{2\backslash j}(n_1), \beta_2(n_1) \mid \tilde{\boldsymbol{\beta}}_1) = 0 \quad \text{for all} \quad n_1 = 0, 1, ..., N_1. \end{aligned} \tag{4.5}$$

Lemma 3. *For some given $c \in \mathbb{R}$, let players in the first stage play the associated monotone strategy profile $\tilde{\boldsymbol{c}}_1$. For player $j \in \mathbb{P}_2$, conditional on its signal b_j and the history h summarized by n_1, the vector $(\beta, \boldsymbol{b}_{2\backslash j})$ is distributed such that:*

(i) The mean of $(\beta, \boldsymbol{b}_{2\backslash j})$ is increasing in b_j;

(ii) The mean of $(\beta, \boldsymbol{b}_{2\backslash j})$ is increasing in n_1 for given c;

(iii) The mean of $(\beta, \boldsymbol{b}_{2\backslash j})$ is increasing in c for given n_1.

Moreover, the conditional covariance matrix Σ_2 of $(\beta, \boldsymbol{b}_{2\backslash j})$ is independent of b_j.

Lemma 4. *(i) $g_1^\varepsilon(\boldsymbol{s}_{1\backslash i}, \tilde{\boldsymbol{c}}_2(), b_i)$ is decreasing in $c_2(n_1)$;*

(ii) $g_1^\varepsilon(\tilde{\boldsymbol{c}}_{1\backslash i}, \tilde{\boldsymbol{c}}_2(), b_i)$ is decreasing in c_1;

(iii) $g_1^\varepsilon(\tilde{\boldsymbol{c}}_{1\backslash i}, \tilde{\boldsymbol{c}}_2(), b_i)$ is increasing in b_i.

Proof. Parts (i)-(iii) are all based on the fact that, by Lemma 3, player i's conditional probability that $b_j > \bar{b}_j$ is increasing in b_i and decreasing in $\bar{b}_j$. *Q.E.D.*

Lemma 5. *XYZ*

(i) $g_2^\varepsilon(n_1, \tilde{\boldsymbol{c}}_{2\backslash j}(n_1), b_j \mid \tilde{\boldsymbol{c}}_1)$ is increasing in c_1;

(ii) $g_2^\varepsilon(n_1, \tilde{\boldsymbol{c}}_{2\backslash j}(n_1), b_j \mid \tilde{\boldsymbol{c}}_1)$ is decreasing in $c_2(n_1)$;

(iii) $g_2^\varepsilon(n_1, \tilde{\boldsymbol{c}}_{2\backslash j}(n_1), b_j \mid \tilde{\boldsymbol{c}}_1)$ is increasing in n_1;

(iv) $g_2^\varepsilon(n_1, \tilde{\boldsymbol{c}}_{2\backslash j}(n_1), b_j \mid \tilde{\boldsymbol{c}}_1)$ is increasing in b_j.

Proposition 4. *The sequential global game always has a perfect Bayesian equilibrium in increasing strategies.*

Proof. [6] Assume that we have monotone strategies and that the payoff functions are continuous in β. First, we have

$$s_i = \begin{cases} 0 & b_i < \beta_i^* \\ 1 & b_i \geq \beta_i^* \end{cases},$$

where $\beta_i^* = \beta_1$ for the first-period players and $\beta_i^* = \beta_2(n_1)$ for the second-period players. For any $d > 0$ and $V \in \mathbb{R}^d$, let I_V be the indicator function for the set V. Then the payoff of any player i in the first period, for the given actions of other players $x_{1/i}, x_2$ and the given parameter β, can be written as

$$G_1(x_{1/i}, x_2, \beta) = \sum_{s \in \{0,1\}^{n-1}} I_{s=(x_{1/i}, x_2)} G_1^s(\beta) = \sum_{s \in \{0,1\}^{n-1}} I_{s \circ (\boldsymbol{b_{/i}} - \beta^*) + (1-s)\circ(\beta^*) - \boldsymbol{b_{/i}})) \geq 0} G_1^s(\beta),$$

where $\circ$ denotes the Hadamard (element-wise) product and $\boldsymbol{b_{/i}}$ is the vector of b for all players but i. As a result, $G_1(x_{1/i}, x_2, \beta)$ is a linear combination of continuous univariate functions of β. Analogously, the payoff of any player j in the second period, for the given actions of other players $h = x_1, x_{2/j}$ and the given parameter β, can be written as

$$G_2(h, x_{2/j}, \beta) = \sum_{s \in \{0,1\}^{n-1}} I_{s=(h, x_{2/j})} G_2^s(\beta) = \sum_{s \in \{0,1\}^{n-1}} I_{s \circ (\boldsymbol{b_{/j}} - \beta^*) + (1-s)\circ(\beta^*) - \boldsymbol{b_{/j}})) \geq 0} G_2^s(\beta).$$

Now, in the equilibrium we have the condition $b_i = \beta_1$ in the first period and the condition $b_j = \beta_2$, $\boldsymbol{b_1} \in \mathcal{B}$ in the second period, where $\boldsymbol{b_i}$ is the vector of the i^{th}-period signals, for $i \in \{1, 2\}$, and $\mathcal{B}$ is defined more or less as in your paper. Then we have

$$\begin{aligned} g_1^{s_{1/i}, s_2}(\beta_1) &= \int_{\boldsymbol{b_{/i}}, \beta} G_1(x_{1/i}, x_2, \beta) dF(\boldsymbol{b}, \beta | b_i = \beta_1) \\ &= \sum_{s \in \{0,1\}^{n-1}} \int_{s \circ (\boldsymbol{b_{/i}} - \beta^*) + (1-s)\circ(\beta^* - \boldsymbol{b_{/i}}) \geq 0, \beta} G_1^s(\beta) f(\boldsymbol{b}, \beta | b_i = \beta_1) d\boldsymbol{b_{/i}} d\beta \\ &= \sum_{s \in \{0,1\}^{n-1}} \int_{\substack{s_{1/i} \circ (\boldsymbol{\varepsilon_{1/i}} - \varepsilon_i) + (1 - s_{1/i}) \circ (\varepsilon_i - \boldsymbol{\varepsilon_{1/i}}) \geq 0 \\ s_2 \circ (\boldsymbol{b_2} - \beta_2(n_1)) + (1-s_2) \circ (\beta_2(n_1) - \boldsymbol{b_2}) \geq 0, \beta_1 - \varepsilon_i}} G_1^s(\beta_1 - \varepsilon_i) d\hat{F}(\boldsymbol{\varepsilon_1}, \boldsymbol{b_2}), \end{aligned}$$

where $s = (s_{1/i}, s_2)$, $d\hat{F}(\boldsymbol{\varepsilon_1}, \boldsymbol{b_2}) = |J| f(\boldsymbol{\varepsilon_1} + \beta_1 - \varepsilon_i, \boldsymbol{b_2}, \beta_1 - \varepsilon_i)$, and $|J|$ is the determinant of the Jacobian matrix of the transformation $(\boldsymbol{b_{1/i}}, \beta) \to \boldsymbol{\varepsilon_1} : \beta = \beta_1 - \varepsilon_i$, $\boldsymbol{b_{1/i}} = \boldsymbol{\varepsilon_{1/i}} + \beta_1 - \varepsilon_i$. Since $d\hat{F}$ and G_1^s are continuous, the expression in the limit of integration

[6] I thank Olga Kuryatnikova for useful suggestions.

is continuous and nothing else depends on β_1, the expected value is continuous in β_1 and depends on this variable only. Now, let's look at the expected payoff in the second period. We have

$$
\begin{aligned}
g_2^{s_1,s_{2/j}}(\beta_2(n_1),\beta_1) &= \int_{\boldsymbol{b_{/j}},\beta} G_2(h,x_{2/j},\beta)dF(\boldsymbol{b},\beta|b_j=\beta_2(n_1),\boldsymbol{b_1}\in\mathcal{B}) \\
&= \sum_{s\in\{0,1\}^{n-n_1-1}} \int_{\substack{h\circ(\boldsymbol{b_1}-\beta_1)+(1-h)\circ(\beta_1-\boldsymbol{b_1})\geq 0 \\ s\circ(\boldsymbol{b_{2/j}}-\beta_2(n_1))+(1-s)\circ(\beta_2(n_1)-\boldsymbol{b_{2/j}})\geq 0,\beta}} G_2^s(\beta)f(\boldsymbol{b},\beta|b_j=\beta_2(n_1),\boldsymbol{b_1}\in\mathcal{B})d\boldsymbol{b_{/j}}d\beta \\
&= \sum_{s\in\{0,1\}^{n-n_1-1}} \int_{\substack{h\circ(\boldsymbol{b_1}-\beta_1)+(1-h)\circ(\beta_1-\boldsymbol{b_1})\geq 0 \\ s\circ(\boldsymbol{\varepsilon_{2/j}}-\varepsilon_j)+(1-s)\circ(\varepsilon_j-\boldsymbol{\varepsilon_{2/j}})\geq 0,\beta_2(n_1)-\varepsilon_j}} G_2^s(\beta_2(n_1)-\varepsilon_j)d\hat{F}(\boldsymbol{\varepsilon_2},\boldsymbol{b_1}),
\end{aligned}
$$

where $d\hat{F}(\boldsymbol{\varepsilon_2},\boldsymbol{b_1}) = |JJ|f(\boldsymbol{b_1},\boldsymbol{\varepsilon_2}+\beta_2(n_1)-\varepsilon_j,\beta_2(n_1)-\varepsilon_j|\boldsymbol{b_1}\in\mathcal{B})$, and $|JJ|$ is the determinant of the Jacobian matrix of the transformation $(\boldsymbol{b_{2/j}},\beta)\to\boldsymbol{\varepsilon_2}:\beta=\beta_2(n_1)-\varepsilon_j$, $\boldsymbol{b_{2/j}}=\boldsymbol{\varepsilon_{2/j}}+\beta_2(n_1)-\varepsilon_j$. Since $d\hat{F}$ and $G_{\boldsymbol{2}}^s$ are continuous in $\beta_2(n_1)$, the expression in the limit of integration is continuous and nothing else depends on $\beta_2(n_1)$, the expected value is continuous in $\beta_2(n_1)$. The expression depends on β_1 too, but we are not interested i the nature of this dependence. This dependence can look quite bad in terns of β_1 since while conditioning on $\mathcal{B}$ we would have to deal with multivariate truncated normal distributions.

Since both expectations are continuous and can be both positive and negative, they can also be zero. *Q.E.D.*

4.3 Iterated Dominance

Lemma 6.

(i) If *players in stage 1 are known to play some strategy profile* $\boldsymbol{s}_1$ *in* $\hat{S}_1=\{\boldsymbol{s}_1\mid\tilde{\boldsymbol{\alpha}}_1^0\leq\boldsymbol{s}_1\leq\tilde{\boldsymbol{\alpha}}_1^1\}$ *for some* $\alpha_1^0,\alpha_1^1\in\mathbb{R}$ *where* $\alpha_1^0\leq\alpha_1^1$, then *there exist real numbers* $\alpha_2^0(n_1),\alpha_2^1(n_1)$ *where* $\alpha_2^l(0)>\alpha_2^l(1)>...>\alpha_2^l(N_1)$, $l=0,1$, *such that the set of conditionally (on* $\hat{S}_1$*) undominated strategy profiles in stage 2 is given by* $\hat{S}_2(\hat{S}_1)=\{\boldsymbol{s}_2(\cdot,n_1)\mid\tilde{\boldsymbol{\alpha}}_2^0(n_1)\leq\boldsymbol{s}_2(\cdot,n_1)\leq\tilde{\boldsymbol{\alpha}}_2^1(n_1)\}$*. The points* $\alpha_2^0(n_1)$ *and* $\alpha_2^0(n_1)$ *solve* $g_2^\varepsilon(n_1,\tilde{\boldsymbol{\alpha}}_2^0,\alpha_2^0(n_1)\mid\tilde{\boldsymbol{\alpha}}_1^1)=g_2^\varepsilon(n_1,\tilde{\boldsymbol{\alpha}}_2^1,\alpha_2^1(n_1)\mid\tilde{\boldsymbol{\alpha}}_1^0)=0$*.*

(ii) If *players in stage 2 are known to play some strategy* $\boldsymbol{s}_2$ *in* $\hat{S}_2=\{\boldsymbol{s}_2(\cdot,n_1)\mid\tilde{\boldsymbol{\alpha}}_2^0(n_1)\leq\boldsymbol{s}_2(\cdot,n_1)\leq\tilde{\boldsymbol{\alpha}}_2^1(n_1)\}$ *such that* $\alpha_2^l(n_1)$ *is a real number and* $\alpha_2^l(0)>\alpha_2^l(1)>...>\alpha_2^l(N_1)$, $l=0,1$, then *there exists a real numbers* α_1^0,α_1^1 *such*

that the set of conditionally (on $\hat{S}_1$) undominated strategies for players in stage 1 is given by $\hat{S}_1(\hat{S}_2) = \{\boldsymbol{s}_1 \mid \tilde{\boldsymbol{\alpha}}_1^0 \leq \boldsymbol{s}_1 \leq \tilde{\boldsymbol{\alpha}}_1^1\}$. The points α_1^0 and α_1^1 solve $g_1^\varepsilon(\tilde{\boldsymbol{\alpha}}_1^0, \tilde{\boldsymbol{\alpha}}_2^1(\cdot), \alpha_1^0) = g_1^\varepsilon(\tilde{\boldsymbol{\alpha}}_1^1, \tilde{\boldsymbol{\alpha}}_2^0(\cdot), \alpha_1^1) = 0$.

Proof. In Appendix. *Q.E.D.*

We start from stage 1. The lowest expected gain to player $i \in \mathbb{P}_1$, given b_i, would result if $\boldsymbol{s}_{1\setminus i}(b_{\neg i}) = \boldsymbol{s}_2(b_{\neg i}, n_1) = 0$ for all $b_{\neg i}$ and all n_1. In that case, we know (by assumption A3) that there exists a point $\beta_{1,1}$ such that $g_1^\varepsilon(0, 0, \beta_{1,1}) = 0$. For any signal $b_i > \beta_{1,1}$, the action $x_i = 1$ is hence strictly dominant. Since this is true for all players in stage 1, player i knows that $\boldsymbol{s}_{1\setminus i} \geq \tilde{\boldsymbol{\beta}}_{1,1\setminus i}$, and this holds for all i. Observe that (1) $g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}, 0, b_i)$ is strictly increasing in b_i and (2) $g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}, 0, \beta_{1,1}) > g_1^\varepsilon(0, 0, \beta_{1,1}) = 0$. From combining (1) and (2) it follows that there exists a point $\beta'_{1,1} < \beta_{1,1}$ such that $g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}, 0, \beta'_{1,1}) = 0$, making $x_i = 1$ strictly dominant for all $b_i > \beta'_{1,1}$. This again is true for all players in stage 1, so each $i \in \mathbb{P}_1$ knows that $\boldsymbol{s}_{1\setminus i} \geq \tilde{\boldsymbol{\beta}}'_{1,1\setminus i}$. Then again we know that $g_1^\varepsilon(\tilde{\boldsymbol{\beta}}'_{1,1\setminus i}, 0, \beta'_{1,1}) > g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}, 0, \beta'_{1,1}) = 0$, and so on. In this way we can construct a monotone sequence $\beta_{1,1} > \beta'_{1,1} > \beta''_{1,1} > ...$, which is defined on $[\beta_0, \beta_1]$ and therefore must converge. Call its limit $\beta_{1,1}^0$. Similarly, the highest possible expected gain to player i in stage 1 is realized when $\boldsymbol{s}_{1\setminus i}(b_{\neg i}) = \boldsymbol{s}_2(b_{\neg i}, n_1) = 1$. In much the same way as before, this results in a sequence $\beta_{0,1} < \beta'_{0,1} < \beta''_{0,1} < ...$, which is monotone increasing on $[\beta_0, \beta_1]$ and therefore converges to a limit we call $\beta_{0,1}^0$. The set of undominated strategies in stage 1 is therefore given by:

$$S_1^0 := \{\boldsymbol{s}_1 \mid \tilde{\boldsymbol{\beta}}_{1,1}^0 \leq \boldsymbol{s}_1 \leq \tilde{\boldsymbol{\beta}}_{0,1}^0\}.$$

Part *(i)* of Lemma 6 can now be applied to S_1^0 to yield the set of (conditionally) undominated strategies in stage 2:

$$S_2^0(S_1^0) := \{\boldsymbol{s}_2 \mid \tilde{\boldsymbol{\beta}}_{1,2}^0 \leq \boldsymbol{s}_2 \leq \tilde{\boldsymbol{\beta}}_{0,2}^0\}.$$

Observe that:

$$g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{0,1\setminus i}^0, \tilde{\boldsymbol{\beta}}_{0,2}^0, \beta_{0,1}^0) < g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{0,1\setminus i}^0, N_2, \beta_{0,1}^0) = 0,$$

and

$$g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}^0, \tilde{\boldsymbol{\beta}}_{1,2}^0, \beta_{1,1}^0) > g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}^0, 0, \beta_{1,1}^0) = 0.$$

This, combined with part *(ii)* of Lemma 6 to find the set conditionally undominated strategies in stage 1:

$$S_1^1(S_2^0) = \{\boldsymbol{s}_1 \mid \tilde{\boldsymbol{\beta}}_{1,1}^1 \leq \boldsymbol{s}_1 \leq \tilde{\boldsymbol{\beta}}_{0,1}^1\},$$

where $\beta_{1,1}^1 < \beta_{1,1}^0$ and $\beta_{0,1}^1 > \beta_{0,1}^0$ and therefore $S_1^1 \subset S_1^0$. One notes that

$$g_1^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{1,2}^0, \beta_{1,2}^0 \mid \tilde{\boldsymbol{\beta}}_{0,1}^1) > g_1^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{1,2}^0, \beta_{1,2}^0 \mid \tilde{\boldsymbol{\beta}}_{0,1}^0) = 0,$$

and

$$g_1^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{0,2}^0, \beta_{0,2}^0 \mid \tilde{\boldsymbol{\beta}}_{1,1}^1) < g_1^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{0,2}^0, \beta_{0,2}^0 \mid \tilde{\boldsymbol{\beta}}_{1,1}^0) = 0.$$

Given S_1^1 and part (ii) of Lemma 6, we obtain set of conditionally undominated strategies in stage 2:

$$S_2^1(S_1^1) = \{\boldsymbol{s}_2 \mid \tilde{\boldsymbol{\beta}}_{1,2}^1 \leq \boldsymbol{s}_2 \leq \tilde{\boldsymbol{\beta}}_{0,2}^1\},$$

where $\beta_{1,2}^1 < \beta_{1,2}^0$ and $\beta_{0,2}^1 > \beta_{0,2}^0$. Given S_2^1, additional strategies in S_1^1 may become dominated. Inductively, define the points $\beta_{0,1}^{k+1}$ and $\beta_{1,1}^{k+1}$ as the solutions to:

$$\min_{\boldsymbol{s}_2 \in S_2^k} g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}^{k+1}, \boldsymbol{s}_2, \beta_{1,1}^{k+1}) = g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1,1\setminus i}^{k+1}, \tilde{\boldsymbol{\beta}}_{1,2}^k(\cdot), \beta_{1,1}^{k+1}) = 0, \tag{4.6}$$

and

$$\max_{\boldsymbol{s}_2 \in S_2^k} g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{0,1\setminus i}^{k+1}, \boldsymbol{s}_2, \beta_{0,1}^{k+1}) = g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{0,1\setminus i}^{k+1}, \tilde{\boldsymbol{\beta}}_{0,2}^k(\cdot), \beta_{0,1}^{k+1}) = 0, \tag{4.7}$$

respectively. With this inductive definition, we can define:

$$S_1^{k+1}(S_2^k) := \{\boldsymbol{s}_1 \mid \tilde{\boldsymbol{\beta}}_{1,1}^{k+1} \leq \boldsymbol{s}_1 \leq \tilde{\boldsymbol{\beta}}_{0,1}^{k+1}\}. \tag{4.8}$$

We similarly define the points $\beta_{0,2}^{k+1}(n_1)$ and $\beta_{1,2}^{k+1}(n_1)$ as the solutions to

$$\max_{\boldsymbol{s}_1 \in S_1^{k+1}} g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{0,2}^{k+1}(n_1), \beta_{0,2}^{k+1}(n_1) \mid \boldsymbol{s}_1) = g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{0,2}^{k+1}(n_1), \beta_{0,2}^{k+1}(n_1) \mid \tilde{\boldsymbol{\beta}}_{1,1}^{k+1}) = 0, \tag{4.9}$$

and

$$\min_{\boldsymbol{s}_1 \in S_1^{k+1}} g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{1,2}^{k+1}(n_1), \beta_{1,2}^{k+1}(n_1) \mid \boldsymbol{s}_1) = g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{0,2}^{k+1}(n_1), \beta_{0,2}^{k+1}(n_1) \mid \tilde{\boldsymbol{\beta}}_{0,1}^{k+1}) = 0, \tag{4.10}$$

respectively, for all n_1. These definitions allow us to define:

$$S_2^{k+1}(S_1^{k+1}) := \{\boldsymbol{s}_1 \mid \tilde{\boldsymbol{\beta}}_{1,2}^{k+1} \leq \boldsymbol{s}_2 \leq \tilde{\boldsymbol{\beta}}_{0,2}^{k+1}\}. \tag{4.11}$$

In the limit as $k \to \infty$, the sets S_1^k and S_2^k converge to

$$S_1^* = \{\boldsymbol{s}_1 \mid \tilde{\boldsymbol{\beta}}_1^* \leq \boldsymbol{s}_1 \leq \tilde{\boldsymbol{\beta}}_1^{**}\} \tag{4.12}$$

and

$$S_2^* = \{\boldsymbol{s}_2(\cdot, n_1) \mid \tilde{\boldsymbol{\beta}}_2^*(n_1) \leq \boldsymbol{s}_2(\cdot, n_1) \leq \tilde{\boldsymbol{\beta}}_2^{**}(n_1)\}. \tag{4.13}$$

Any strategy profile $\boldsymbol{s}_t \in S_t^*$, $t = 1, 2$, survives iterated elimination of strictly dominated strategies. By construction, the limit points $\beta_1^*, \beta_1^{**}, \beta_2^*, \beta_2^{**}$ simultaneously solve:

$$\begin{aligned}
&g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{2\setminus j}^*(n_1), \beta_2^*(n_1) \mid \tilde{\boldsymbol{\beta}}_1^{**}) = 0 \quad \text{for all } n_1 \\
&g_2^\varepsilon(n_1, \tilde{\boldsymbol{\beta}}_{2\setminus j}^{**}(n_1), \beta_2^{**}(n_1) \mid \tilde{\boldsymbol{\beta}}_1^{*}) = 0 \quad \text{for all } n_1 \\
&g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1\setminus i}^*, \tilde{\boldsymbol{\beta}}_2^*(), \beta_1^*) = 0 \\
&g_1^\varepsilon(\tilde{\boldsymbol{\beta}}_{1\setminus i}^{**}, \tilde{\boldsymbol{\beta}}_2^{**}(), \beta_1^{**}) = 0.
\end{aligned} \tag{4.14}$$

4.4 Vanishing noise

Proposition 5. *Let $\sigma_\varepsilon \to 0$. For some real number $c \in \mathbb{R}$, let first-stage players play the associated monotone strategy profile $\tilde{\boldsymbol{c}}_1$. Given n_1 and b_j, the conditional posterior on $(\beta, \boldsymbol{b}_{2\setminus j})$ of player j is multivariate Normally distributed with mean vector $\hat{\boldsymbol{\beta}}_j = (\hat{\beta}_j, \hat{\beta}_j, ..., \hat{\beta}_j)$ and $\hat{\beta}_j$ given by*

$$\hat{\beta}_j = \begin{cases} \frac{b_j + n_1 \cdot c}{1 + n_1} & \text{if} \quad b_j \leq c \\ \frac{b_j + (N_1 - n_1) \cdot c}{1 + N_1 - n_1} & \text{if} \quad b_j \geq c \end{cases}. \tag{4.15}$$

Importantly, note that $\hat{\beta}_j$ is linear in b_j and c. The conditional covariance matrix of $(\beta, \boldsymbol{b}_{2\setminus j})$ is independent of both b_j and c.

Unlike the one-shot game, second-stage players can no longer assume that others in stage 2 observe a signal either above or below their own with equal probability. This is a consequence of Proposition 5 and, as we shall illustrate shortly, causes substantial complications. First, however, we state a positive result.

Proposition 6. *Let σ_ε be sufficiently small. Sequential global games for which $N_1 \geq 1$ and $N_2 = 1$, i.e. games with any number of first-stage players but only one second-stage player, have a unique perfect Bayesian equilibrium that survives iterated dominance. It is in increasing strategies.*

Proof. Suppose there are two equilibria in increasing strategies, $(\tilde{\boldsymbol{\beta}}_1^0, \tilde{\boldsymbol{\beta}}_2^0())$ and $(\tilde{\boldsymbol{\beta}}_1^1, \tilde{\boldsymbol{\beta}}_2^1())$. Without loss of generality, let $\beta_1^1 > \beta_1^0$. By Lemma 1, we know that player i in stage 1 who observes $b_i = \beta_1^0$ then believes that every other player's signal is either below or above β_1^0 with probability 1/2. Similarly, if player i observes $b_i = \beta_1^1$, he believes that every other player's signal is either below or above β_1^1 with probability 1/2. It is easy to see, then, that $\beta_1^1 > \beta_1^0$ can be consistent with equilibrium for players in stage 1 if and only if $\beta_2^1(n_1) > \beta_2^0(n_1)$ for at least one n_1. But for n_1 and b_j, the posterior of player j in stage 2 on β will be higher when n_1 is the realization of $\tilde{\boldsymbol{\beta}}_1^1$ compared to when it is the realization of $\tilde{\boldsymbol{\beta}}_1^0$. As a consequence, $(\tilde{\boldsymbol{\beta}}_1^0, \tilde{\boldsymbol{\beta}}_2^0())$ and $(\tilde{\boldsymbol{\beta}}_1^1, \tilde{\boldsymbol{\beta}}_2^1())$ can be equilibria for player j in stage 2 if and only if $\beta_2^1(n_1) < \beta_2^0(n_1)$ for all n_1. It follows that $(\tilde{\boldsymbol{\beta}}_1^0, \tilde{\boldsymbol{\beta}}_2^0())$ and $(\tilde{\boldsymbol{\beta}}_1^1, \tilde{\boldsymbol{\beta}}_2^1())$ cannot be separate equilibria. But then, looking at (4.14), it must be the case that $\beta_1^* = \beta_2^*$ and $\beta_2^*(n_1) = \beta_2^{**}(n_1)$ for all n_1, establishing the result. *Q.E.D.*

Theorem 1. *The sequential global game does not generally have a unique equilibrium surviving iterated dominance.*

4.5 An Example

Consider a simple game with one first-mover and two (simultaneous) second-movers. Suppose player 1 plays the monotone strategy $\tilde{\beta}_1^0$, i.e.there is a point β_1^0 such that player 1 plays $x_1 = 0$ for all signals $b_1 < \beta_1^0$ while he plays $x_1 = 1$ for all $b_1 > \beta_1^0$.

We focus for now on the case where player 1 has played $x_1 = 1$ and $\sigma_\varepsilon \to 0$. Since players 2 and 3 in the second stage observe the history of play, they know that $b_1 \geq \beta_1^0$. Hence, if player 2 in stage 2 observes the signal $b_2 < \beta_1^0$, his posterior on β will be $\hat{\beta} = (b_2 + \beta_1^0)/2 > b_2$, see Proposition 5. On the other hand, if he observes a signal $b_2 > \beta_1^0$, his posterior on β is simply $\hat{\beta}_2 = b_2$. Figure 1 illustrates.

From figure 1 it follows immediately that player 2's posterior on β is strictly increasing in b_2. Taking the action of player 3 as given, this would imply that 2's expected gain is strictly increasing in b_2 as well. However, we cannot take x_3 as given; it will depend on his signal b_3. Since we are interested in symmetric increasing strategies anyway, suppose now that player 3 follows the strategy $\tilde{\beta}_2$ for some β_2. What we want to know is player 2's posterior on x_3, the action of player 2, when he observes exactly $b_2 = \beta_2$, player 3's switching point. Figure 2 plots this posterior.

When $\beta_2 < \beta_1^0$ and $b_2 = \beta_2$, player 2 thinks that the true β is $(\beta_1^0 + \beta_2)/2 > \beta_2$. He therefore believes that player 3 will observe $b_3 > \beta_2$ and therefore that $x_3 = 1$ with probability 1. On the other hand, if $\beta_2 > \beta_1^0$ and player 2 observes $b_2 = \beta_2$, he thinks

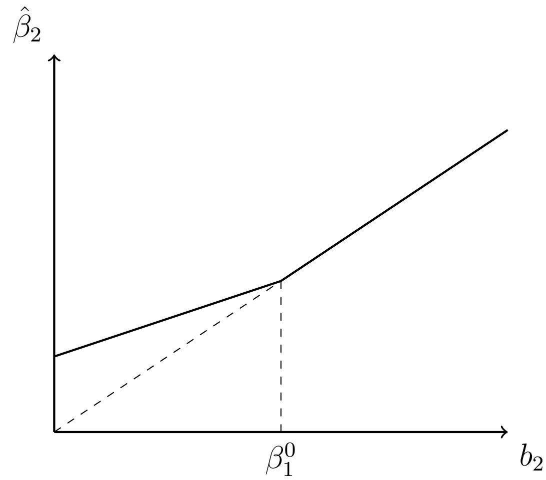

Figure 1: Player 2's posterior on β.

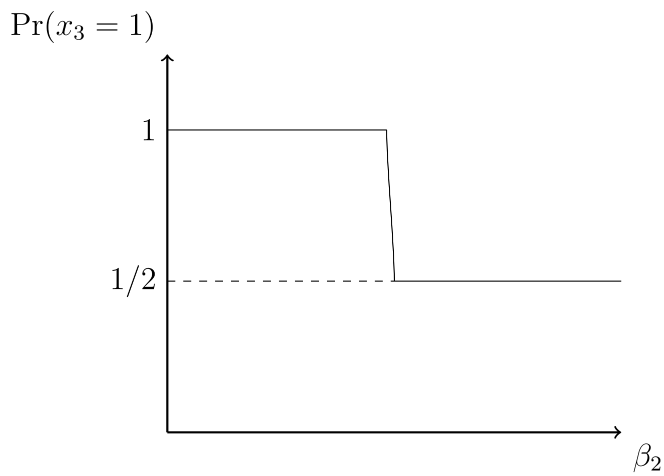

Figure 2: Player 2's posterior posterior probability that $x_3 = 1$ when player 2 observes signal $b_2 = \beta_2$ and player 3 plays $\tilde{\beta}_2$.

that the true β is simply $b_2 = \beta_2$. His posterior on b_3 is therefore that $b_3 > \beta_2$ with probability 1/2, which means that he believes the probability with which player 3 will play $x_3 = 1$ is also 1/2. This has an important implications. For $b_2 = \beta_2$ very close to but below β_1^0, player 2's posterior on β is increasing in his signal indeed, but his posterior on x_3 is *decreasing.*

The previous observation is crucial. It implies that for $\beta_2 = b_2$ *very close* to β_1^0, player 2's expected gain is decreasing in $\beta_2 = b_2$: even though his posterior on β will still be increasing, his posterior probability on x_3 is decreasing much more rapidly

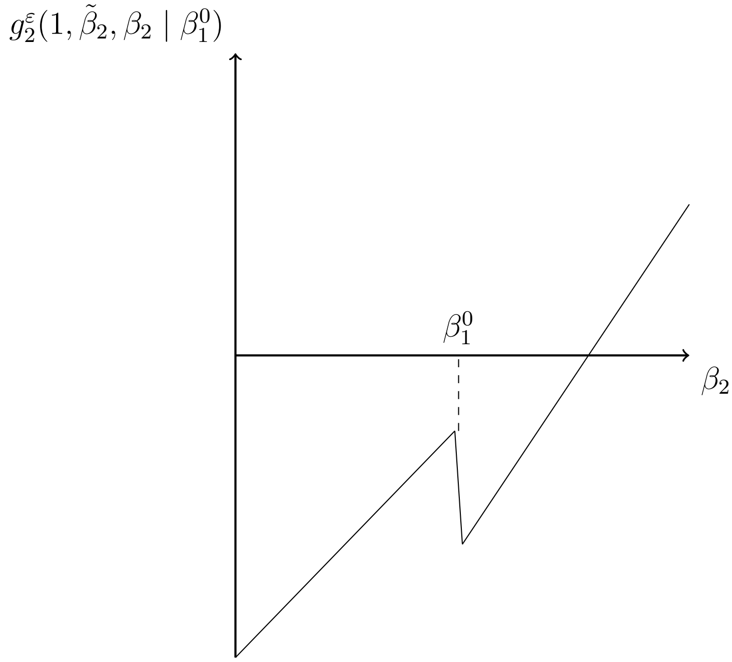

Figure 3: Example of $g_2^\varepsilon()$ where the decreasing part of the curve falls entirely in the domain for which it is negative.

making it impossible that the marginally higher posterior on β compensates for the radical drop in his expectations of x_3.

Plotting the expected gain function $g_2^\varepsilon(1, \tilde{\beta}_2, \beta_2 \mid \beta_1^0)$, we therefore observe the following. Except for some very small interval around β_1^0, $g_2^\varepsilon(1, \tilde{\beta}_2, \beta_2 \mid \beta_1^0)$ is strictly increasing in β_2. There are three possible scenarios for this "continuous blip" in $g_2^\varepsilon(1, \tilde{\beta}_2, \beta_2 \mid \beta_1^0)$. First, it may fall entirely in a region where g_2^ε is negative. This case is illustrated in figure 4.5. Second, it may fall entirely in the region for which g_2^ε is positive; see figure 4.5. Finally, $g_2^\varepsilon(1, \tilde{\beta}_2, \beta_2 \mid \beta_1^0)$ may be positive when β_2 is just below β_1^0 and then fall below zero as β_2 increases due to the expected drop in x_3. In this case, the game may not have a unique equilibrium.

5 Discussion and Conclusions

This paper introduces sequential global games. We obtain several notable results. First, a sequential global game always has at least one equilibrium in monotone strategies. This is an existence result and does not exclude there may be more than one, or equilibria in other types of strategies. Importantly, while the set of equilibria may also

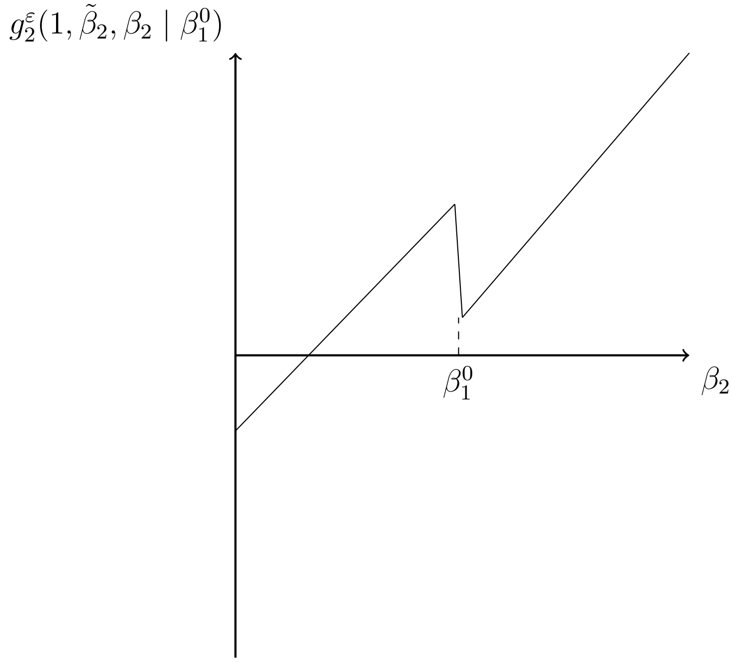

Figure 4: Example of $g_2^{\varepsilon}()$ where the decreasing part of the curve falls entirely in the domain for which it is positive.

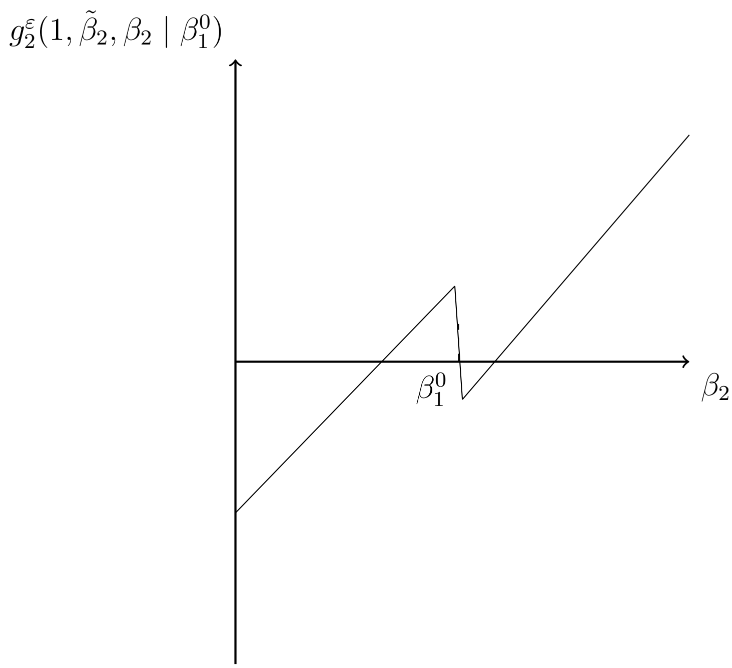

Figure 5: Example of $g_2^{\varepsilon}()$ where the decreasing part of the curve starts int he domain for which it is positive but ends in the domain for which it is negative.

be larger in one-shot global games, it is generally true that this set reduces to a point if we let the noise in players' private signals become arbitrarily small. This result breaks down in a sequential global game. Even a simple two-stage game is not guaranteed to have a unique equilibrium when the noise vanishes. My result demonstrates the importance of carefully considering the type of dynamic one studies in applied work on global games.

From a theoretical point of view, it is interesting the set of equilibria in a sequential global game may be larger than the set of equilibria in one-shot global games. For games of perfect information, the opposite is true: if we take some one-shot game and transform it into a sequential game by splitting the player set into two (not necessarily singleton) subsets, the set of subgame perfect equilibria in the latter game is weakly smaller than the set of Nash equilibria of the original one-shot game. On the other hand, the set of perfect Bayesian equilibria of the sequential global game may be larger than the set of Bayesian Nash equilibria of the one-shot game.

An interesting avenue for future research are sequential global games where the order of players is endogenous. In the present work, the player set was partitioned in an arbitrary and exogenous way. While there are no clear objections to such an approach from an abstract theory perspective, in applications it may matter. When cellphone were first introduced, there did not exist an exogenous black box spitting out who could buy the first generation of phones and who had to wait for the second. Rather, we may well imagine this was driven both by varying preferences among individuals and their differing beliefs regarding the usefulness of mobile phones. The latter case can be directly translated into global games language as differences in signals received at the start of the game.

The theoretical predictions made in simultaneous move global games are borne out in the laboratory, see in particular Heinemann et al. (2004, 2009). It would be interesting to see whether sequential global games are equally successful when put to test.

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A Proofs & Derivations

A.1 Proofs for Section 3

Lemma 7. *For all $i \in \mathbb{P}$, for given $c \in \mathbb{R}$ and associated monotone strategy profile $\tilde{\boldsymbol{c}}$,*

$$g^{\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, b_i) > g^{\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, b_i') \iff b_i > b_i'. \tag{A.1}$$

Moreover, for any two $c, d \in \mathbb{R}$ and associated monotone strategy profiles $\tilde{\boldsymbol{c}}_{\neg i}$ and $\tilde{\boldsymbol{d}}_{\neg i}$, for all b_i,

$$g^{\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, b_i) > g^{\varepsilon}(\tilde{\boldsymbol{d}}_{\neg i}, b_i) \iff d > c. \tag{A.2}$$

Proof. Follows immediately from Lemma 1 in combination with assumptions (A1) and (A2).

Proof of Lemma 2

Proof. Let $\sigma_\varepsilon^2 \to 0$. Then player i's conditional distribution on $\boldsymbol{b}_{\neg i}$ is multivariate normal with mean vector $(b_i, b_i, ..., b_i)$. Hence, conditional on b_i player i believes that $\Pr[b_j > b_i \mid b_i] = \Pr[b_j < b_i \mid b_i] = 1/2$. The conditional distribution of $\boldsymbol{x}_{\neg i}$ is therefore equivalent whether:

(i) All players $j \neq i$ play a monotone strategy with switching point c and player i receives signal $b_i = c$;

(ii) All players $j \neq i$ play a monotone strategy with switching point d and player i receives signal $b_i = d$.

Since, conditional on n, the gain g is strictly increasing in b_i, we conclude that $g^\varepsilon(\tilde{\boldsymbol{c}}_{\neg i}, c) > g^\varepsilon(\tilde{\boldsymbol{d}}_{\neg i}, d) \iff c > d$. As this inequality is strict when $\sigma_\varepsilon \to 0$, we can allow $\sigma_\varepsilon > 0$ and the result is still correct. *Q.E.D.*

Proof of Proposition 2

Proof. By definition, β_0 and β_1 solve

$$g(\boldsymbol{t}_{\neg i}(\cdot; -\boldsymbol{\infty}), \beta_0) = 0, \tag{A.3}$$

and

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\infty}), \beta_1) = 0, \tag{A.4}$$

respectively. Define $\boldsymbol{\beta}_1 = (\beta_1, \beta_1, .., \beta_1)$ and $\boldsymbol{\beta}_0 = (\beta_0, \beta_0, .., \beta_0)$.

Any rational player i will neither play $s_i(b_i) = 1$ when $b_i < \beta_0$, nor $s_i(b_i) = 0$ when $b_i > \beta_1$, for such strategies are strictly dominated. The remaining, undominated strategies then satisfy the following inequalities: $t_i(b_i; \beta_0) \leq s_i(b_i) \leq t_i(b_i; \beta_1)$, for all $b_i \in \mathbb{R}$. Letting $S_i^1 \subseteq S_i$ denote the subset of these undominated strategies in S_i, one can conclude that player i will only play a strategy from $S_i^1 := \{s_i : \forall b_i \in \mathbb{R} : t_i(b_i; \beta_1) \leq s_i(b_i) \leq t_i(b_i; \beta_0)\}$, for all i. Out of completeness, define $S_{\neg i}^1 := \prod_{j \neq i} S_j^1$ and $S^1 := S_{\neg i}^1 \times S_i^1$, where a strategy-profile $s \notin S^1$ prescribes strictly dominated behavior to at least one player. Since strictly dominated strategies can be disregarded, players effectively play the *reduced game* which is the original game but with all strictly dominated strategies removed from the set of strategy profiles.

In the reduced game under S^1, too, g exhibits strategic complementarity. The highest expected gain to player i then realizes if all other players j play $s_j(b_j) = 1$

unless this is strictly dominated, which it is for all $b_j < \beta_0$. Hence, player i's highest gain obtains from the deleted strategy profile $\boldsymbol{t}_{\neg i}(\cdot;\boldsymbol{\beta}_0)$, for see the definition of t in (**??**). Similarly, player i's lowest expected gain results if the other players play $t_{\neg i}(\cdot;\beta_{1,\neg i})$.[7] In the notation, for any b_i player i's highest gain, conditional on all other players not playing a strictly dominated strategy, is given by $g((t_{\neg i}(\cdot,\beta_0)),b_i)$, the lowest by $g((t_{\neg i}(\cdot,\beta_1)),b_i)$.

Now define β_0^1 as the point that solves:

$$g(\boldsymbol{t}_{\neg i}(\cdot;\boldsymbol{\beta}_0),\beta_0^1) = 0. \tag{A.5}$$

Similarly, define β_1^1 as the solution to:

$$g(\boldsymbol{t}_{\neg i}(\cdot;\boldsymbol{\beta}_1),\beta_1^1) = 0. \tag{A.6}$$

Note that $g(t_{\neg i}(\cdot;-\infty),\beta_0) = 0$ and $\beta_0 > -\infty$. From that, it follows that $g(\boldsymbol{t}_{\neg i}(\cdot;-\infty),b_i) < g(\boldsymbol{t}_{\neg i}(\cdot;\boldsymbol{\beta}_0),b_i)$ for any b_i. As β_0^1 solves (A.5), one concludes that $\beta_0^1 > \beta_0$. It can similarly be demonstrated that $\beta_1^1 < \beta_1$.

Hence, if player i knows that no player j will play a strictly dominated strategy, this means player i will expect a strictly negative gain from playing $x_i = 1$ for all $b_i < \beta_0^1$ or from playing $x_i = 0$ for all $b_i > \beta_1^1$. It follows that player i, knowing that no player j plays a strictly dominated strategy, will only play a strategy from $S_i^2 \subseteq S_i^1$, where $S_i^2 = \{s_i : \forall b_i \in \mathbb{R} : t_i(b_i;\beta_1^1) \leq s_i(b_i) \leq t_i(b_i;\beta_0^1)\}$.

One can repeat this procedure for any arbitrary number k of times, inductively defining β_0^{k+1} and β_1^{k+1} as the points that solve:

$$g(\boldsymbol{t}_{\neg i}(\cdot;\boldsymbol{\beta}_0^k),\beta_0^{k+1}) = g(\boldsymbol{t}_{\neg i}(\cdot;\boldsymbol{\beta}_1^k),\beta_1^{k+1}) = 0. \tag{A.7}$$

If all players are rational and this rationality is common knowledge, no player i will therefore play a strategy not belonging to S_i^{k+1}, defined as:

$$S_i^{k+1} := \left\{s_i : \forall b_i \in \mathbb{R} : t_i(b_i;\beta_1^k) \leq s_i(b_i) \leq t_i(b_i;\beta_0^k)\right\}. \tag{A.8}$$

Which set of iteratively undominated strategies obtains if one repeats this process on and on? The following lemma will help answering that question.

[7] We mean to say that $t_{\neg i}(b_{\neg i};\beta_0) = \sup_{s_{\neg i}\in S_{\neg i}^1} g(s_{\neg i},b_i)$ and $t_{\neg i}(b_{\neg i};\beta_1) = \inf_{s_{\neg i}\in S_{\neg i}^1} g(s_{\neg i},b_i)$.

Lemma 8. *For each player i in $\mathbb{P}$,*

(i) The action $x_i = 0$ is iteratively dominant at all $b_i < \beta_0^* \in (\beta_0, \beta_1)$, where β_0^* is the limit of the sequence $(\beta_0^k)_{k=0}^{\infty}$.

(ii) The action $x_i = 1$ is iteratively dominant at all $b_i > \beta_1^* \in (\beta_0, \beta_1)$, where β_1^* is the limit of the sequence $(\beta_1^k)_{k=0}^{\infty}$.

Proof. $g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{c}), b_i)$ is monotone increasing in b_i, and monotone decreasing in c. Moreover, $\beta_0 > -\infty$. Hence, if β_0 $(= \beta_0^0)$ solves $g(\boldsymbol{t}_{\neg i}(\cdot; -\boldsymbol{\infty}), \beta_0^0) = 0$ while β_0^1 solves $g(t_{\neg i}(\cdot; \boldsymbol{\beta}_0), \beta_0^1) = 0$, it must be that $\beta_0^0 < \beta_0^1$. By induction on this argument, it follows that $\beta_0^{k+1} > \beta_0^k$, for all k. Therefore, $(\beta_0^k)_{k=0}^{\infty}$ is a monotone increasing sequence. Any monotone sequence defined on a compact set (the interval $[\beta_0, \beta_1]$ is compact) converges to a point in the set. Hence, $(\beta_0^k)_{k=0}^{\infty}$ indeed has a limit and we label it β_0^*. Similarly, $(\beta_1^k)_{k=0}^{\infty}$ is a monotone (decreasing) sequence, which therefore has a limit, called β_1^*. *Q.E.D.*

Since $(\beta_0^k)_{k=0}^{\infty}$ and $(\beta_1^k)_{k=0}^{\infty}$ are converging, consecutive terms in either sequence become arbitrarily close to each other as $k \to \infty$.[8] Moreover, since conditional on $\boldsymbol{\beta}_0^k$ and $\boldsymbol{\beta}_1^k$, the points β_0^{k+1} and β_1^{k+1} are defined as the solution to (A.7), the limits β_0^* and β_1^* are characterized by:

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_0^*), \beta_0^*) = g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_1^*), \beta_1^*) = 0, \tag{A.9}$$

as given in the proposition. *Q.E.D.*

Proof of Lemma ??

Proof. By assumptions A2 and A3, $G_1(x_2; \beta)$ is increasing in x_2 and β. Hence, given b_1 and two strategies s_2 and s_2' where $s_2(h, b_2) \geq s_2'(h, b_2)$ for all h and all b_2, we have $\iint G_1(s_2(h, b_2), \beta)\, dF^{\varepsilon}(\beta, b_2 \mid b_1) \geq \iint G_1(s_2'(h, b_2), \beta)\, dF^{\varepsilon}(\beta, b_2 \mid b_1)$, where the inequality is strict if $s_2(h, b_2) > s_2'(h, b_2)$ for at least one b_2 and h. Hence, if we specify the strategies s_2 and s_2' as follows:

$$s_2(h, b_2) = \begin{cases} t_2(b_2; \bar{b}^0) & \text{if} \quad h = 0 \\ t_2(b_2; \bar{b}^1) & \text{if} \quad h = 1 \end{cases},$$

[8]That is, for any real number $\nu > 0$, one can find a K_l such that $|\beta_l^{k+1} - \beta_l^k| < \nu$ for all $k \geq K_l$, for $l = 0, 1$.

and

$$s_2'(h, b_2) = \begin{cases} t_2(b_2; \bar{b}^0) & \text{if} \quad h = 0 \\ t_2(b_2; \bar{b}^{1'}) & \text{if} \quad h = 1 \end{cases},$$

we observe that $\iint G_1(s_2(h, b_2), \beta)\, dF^\varepsilon(\beta, b_2 \mid b_1) > \iint G_1(s_2'(h, b_2), \beta)\, dF^\varepsilon(\beta, b_2 \mid b_1)$ if and only of $\bar{b}^1 > \bar{b}^{1'}$. This proves part *(i)*. Part *(ii)* is proven in a symmetric way. *Q.E.D.*

Proof of Lemma 5

Proof. For given $\boldsymbol{b}_{2\setminus j}$ and b_j, part (i) is an immediate implication of the fact that j's posterior on $(\beta, \boldsymbol{b}_{2\setminus j})$ induced by $\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{b}}_1)$ lies to the right to the posterior on $(\beta, \boldsymbol{b}_{2\setminus j})$ induced by $\boldsymbol{t}_1(\cdot; \bar{\bar{\boldsymbol{b}}}_1)$, for any history h, if and only if $\bar{\boldsymbol{b}}_1 \geq \bar{\bar{\boldsymbol{b}}}_1$ (see Lemma 3 part (ii)). Since, ceteris paribus, G_j is increasing in both $\boldsymbol{x}_{2\setminus j}$ and β, this establishes part (i) of the lemma.

Given h, given the first-stage strategy profile $\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{b}}_1)$, and given b_j, player j's conditional posterior distribution on $(\beta, \boldsymbol{b}_{2\setminus j})$ is given. For any posterior on β, the probability that $\boldsymbol{b}_{2\setminus j} > \bar{\boldsymbol{b}}_{2\setminus j}$ is greater than the probability that $\boldsymbol{b}_{2\setminus j} > \bar{\bar{\boldsymbol{b}}}_{2\setminus j}$ if and only if $\bar{\bar{\boldsymbol{b}}}_{2\setminus j} > \bar{\boldsymbol{b}}_{2\setminus j}$. Since, given $h = \boldsymbol{x}_1$ and β, G_j is increasing in $\boldsymbol{x}_{2\setminus j}$, this establishes part (ii) of the lemma.

Given h and the first-stage strategy profile $\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{b}}_1)$, player j's conditional posterior distribution on $(\beta, \boldsymbol{b}_{2\setminus j})$ is first-order stochastically in increasing b_j (see Lemma 3 part (i)). Since, ceteris paribus, G_j is increasing in both $\boldsymbol{x}_{2\setminus j}$ and β, this establishes part (iii) of the lemma. *Q.E.D.*

A.2 Proofs for Section 4

Proof of Lemma 3

Proof. Since h is the realization of $\tilde{\boldsymbol{c}}_1$, player j learns that n_1 players $i \in \mathbb{P}_1$ must have observed a signal $b_1 \geq c$ while $N_1 - n_1$ players i received $b_1 \leq c$. The likelihood of observing the signal b_j and the history h under strategy profile $\tilde{\boldsymbol{c}}_1$, as a function of β, is

$$L(\beta) = \left[1 - \Phi\left(\frac{c - \beta}{\sigma_\varepsilon}\right)\right]^{n_1} \cdot \left[\Phi\left(\frac{c - \beta}{\sigma_\varepsilon}\right)\right]^{N_1 - n_1} \cdot \phi\left(\frac{b_j - \beta}{\sigma_\varepsilon}\right) \cdot \phi\left(\frac{\beta - \bar{\beta}}{\sigma_\beta}\right). \quad \text{(A.10)}$$

The point $\hat{\beta}_j$ that maximizes the likelihood function L gives us the mean of player j's posterior (Normal) distribution on β.

Define $m(x) := \phi(x)/(1-\Phi(x))$. Define $\xi = (c-\beta)/\sigma_\varepsilon$. Taking the natural logarithm of (A.10) and differentiating with respect to β, we can rewrite:

$$\frac{\partial}{\partial\beta}\ln(L(\beta)) = \frac{n_1}{\sigma_\varepsilon}m(\xi) - \frac{N_1-n_1}{\sigma_\varepsilon}m(-\xi) + \frac{b_j-\beta}{\sigma_\varepsilon^2} - \frac{\beta-\bar{\beta}}{\sigma_\beta^2}.$$

Let $\hat{\beta}_j$ solve $L'(\hat{\beta}_j) = 0$, our ML-estimator of β. Multiply by σ_ε^2 to obtain:

$$n_1 \cdot m(\hat{\xi}_j)\sigma_\varepsilon - (N_1-n_1)\cdot m(-\hat{\xi}_j)\sigma_\varepsilon + (b_j-\hat{\beta}_j) - \frac{\sigma_\varepsilon^2}{\sigma_\beta^2}(\hat{\beta}_j-\bar{\beta}) = 0,$$

where $\hat{\xi}_j = (c-\hat{\beta}_j)/\sigma_\varepsilon$. We thus obtain:

$$\lambda b_j + (1-\lambda)\bar{\beta} = \hat{\beta}_j - \lambda n_1 \cdot m(\hat{\xi}_j)\sigma_\varepsilon + \lambda(N_1-n_1)\cdot m(-\hat{\xi}_j)\sigma_\varepsilon.$$

where $\lambda = \sigma_\beta^2/(\sigma_\beta^2+\sigma_\varepsilon^2)$. We therefore know that $\hat{\beta}_j$ is (i) increasing in b_j, (ii) increasing in n_1, and (iii) increasing in c. *Q.E.D.*

Proof of Lemma 5

Proof. Recall from the proof of Lemma 3 that the expected value $\hat{\beta}_j$ is implicitly defined by:

$$n_1 \cdot m(\hat{\xi}_j)\sigma_\varepsilon - (N_1-n_1)\cdot m(-\hat{\xi}_j)\sigma_\varepsilon + (b_j-\hat{\beta}_j) - \frac{\sigma_\varepsilon^2}{\sigma_\beta^2}(\hat{\beta}_j-\bar{\beta}) = 0,$$

where $m(x) = \phi(x)/(1-\Phi(x))$ and $\xi_j = (c-\hat{\beta}_j)/\sigma_\varepsilon$. Observe that $\sigma_\varepsilon \to 0$ implies $\xi \to \pm\infty$. We know that $m(x)/x \to 1$ if $x \to +\infty$ and $m(x)/x \to 0$ for $x \to -\infty$. Finally, since $\hat{\xi} = (c-\hat{\beta})/\sigma_\varepsilon$, we note that $m(\hat{\xi})\sigma_\varepsilon$ can be rewritten as $(m(\hat{\xi})/\hat{\xi})(c-\hat{\beta})$. Solving for $\hat{\beta}$ yields the result. *Q.E.D.*