Sequential Global Games

Roweno J.R.K. Heijmans*

Tilburg University

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October 14, 2020

Abstract

Global games are incomplete information games where players receive private noisy signals about the true game played. In a sequential global game, the set of players is partitioned into subsets. Players within a subset (of the partition) play simultaneously but no two subsets move at the same time. The resulting sequence of stages introduces intricate dynamics not encountered in simultaneous move global games. We show that a sequential global game with strategic complementarities and binary actions has at least one equilibrium in monotone strategies. When signals are sufficiently precise, the sequential global game has a unique equilibrium satisfying iterated dominance, forward induction, and backward induction, even if the complete information game (given the partition of the player set) has multiple equilibria. Several applications are discussed.

1 Introduction

Global games are a class of incomplete information games where players receive private noisy signals about the true game being played. Introduced by Carlsson and Van Damme (1993), the approach has been applied to such wide-ranging phenomena as currency

*I wish to thank Eric van Damme and Reyer Gerlagh, my supervisors, for invaluable comments.

attacks (Morris and Shin, 1998), regime switches (Chamley, 1999; Angeletos et al., 2007), financial crises (Angeletos and Werning, 2006), political protests (Edmond, 2013), bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005), and platform markets Jullien and Pavan (2019). This paper introduces sequential global games with strategic complementarities and binary actions.

In a sequential global game, the player set is partitioned into (not necessarily singleton) subsets. Once the true game is drawn and private signals are received, players choose their actions. While players within a subset play simultaneously, no two subsets move at the same time. Instead, different subsets move in a given order of stages and each subset moves only once. In any stage, the history of play in earlier stages is observed. Individual payoffs depend on the true game drawn and the actions chosen by all players.

The timing of sequential global games is different from that studied in simultaneous move (Carlsson and Van Damme, 1993; Frankel et al., 2003) or repeated (Angeletos et al., 2007) global games. In a simultaneous move game, all players move once and simultaneously after receiving their signals. In a repeated game, all players move simultaneously as well but they do so multiple times, and each stage game is drawn separately.

We restrict attention to sequential global games with strategic complementarities (Bulow et al., 1985) and binary actions. Games of regime change are an often-studied special case of games in this class, applications including currency attacks (Morris and Shin, 1998), regime switches (Chamley, 1999; Angeletos et al., 2007), financial crises (Angeletos and Werning, 2006), political protests (Edmond, 2013), and bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005). Strategic complementarities cut to the core of equilibrium multiplicity and coordination failure in complete information coordination games (Cooper and John, 1988; Van Huyck et al., 1990). By adding uncertainty about the true game played, we prove that sequential global games with strategic complementarities have a unique equilibrium satisfying iterated dominance, backward induction, and forward induction if private signals are sufficiently precise.

Sequential global games have intricate dynamics not encountered in simultaneous move environments. Consider a two-stage game of staggered investments in some novel network technology, where uncertainty and signals pertain to the technology's usefulness or quality.[1] At the start of stage 2, firms observe the investments made

[1]The Electronic Medical Record would be an example of such a technology, see Dranove et al. (2014).

in stage 1. Not only does this observation establish a minimum network size for the technology (stimulating stage 2 investments), it also provides indirect evidence of the technology's quality. In equilibrium, these effects are mutually reinforcing. Firms in stage 1 will adopt the technology only if its quality is perceived to be high. In stage 2, large first-stage investments are then good news for two reasons: (i) there is a sizable network to join by investing, and (ii) the technology's true quality is likely to be high. Firms in stage 2 will therefore invest even for relatively low private quality assessments (the large network and high quality assessment of first-stage firms make up for low private estimates). The readiness of stage 2 firms to invest in turn affects investment in the first stage. This leads to an intricate cycle where investment decisions in different stages influence each other back and forth.

It is well known that simultaneous move global games can select a Pareto dominated equilibrium. The same is true for sequential global games. However, for a range of true games the sequential global game selects an efficient equilibrium whereas the simultaneous move global game does not (the comparison is for a given players set, clearly). If the interaction between agents is subject to uncertainty and individual choices are strategic complements, the planner may prefer to let groups of agents interact sequentially. Consider climate change, a profoundly uncertain process avoidable by large-scale reductions in greenhouse gas emissions. Emission reductions are achieved if countries install green technologies, investments in which can be considered strategic complements (c.f. Karp, 2017). While global high-profile meetings have traditionally been relied upon to ratify international environmental agreements such as the Paris Agreement, our theory suggests global welfare may be enhanced if groups of countries are let to decide in a prescribed order.[2]

After our theoretical explorations, we discuss several applications. These include network externalities (Katz and Shapiro, 1985), international environmental agreements (Barrett, 1994; Harstad, 2012, 2016; Battaglini and Harstad, 2016), games of regime change (Morris and Shin, 1998; Angeletos et al., 2007; Edmond, 2013), financial and housing markets (Ganguli and Yang, 2009; Manzano and Vives, 2011), and global disease eradication (Barrett, 2003). While these applications have been studied before, we argue all of them possess characteristics that motivate a sequential global games approach. Our theory may broaden understanding of these phenomena.

[2]The statement pertains to global welfare. While each country will (in expectations) benefit from sequential play, depending on the order of moves some countries may be still better off than others. This creates an array of political economy complications not studied in the paper.

2 The Game

Let the set of players be $\mathbb{P} = \{1, 2, ..., N\}$. Each agent $i \in \mathbb{P}$ chooses action $x_i \in \{0, 1\}$. We define $\boldsymbol{x} := (x_i)_{i\in\mathbb{P}}$. The vector of actions played by all players but i is:

$$\boldsymbol{x}_{\neg i} := (x_j)_{j\in\mathbb{P}\setminus\{i\}}. \tag{2.1}$$

If player i plays x_i while the other players play $\boldsymbol{x}_{\neg i}$, the payoff to player i is given by $u(x_i, \boldsymbol{x}_{\neg i}, \beta)$, where β is a payoff-relevant parameter. The payoff function u is symmetric with respect to each x_j in $\boldsymbol{x}_{\neg i}$ and the same for all players i.[3] Due to the symmetry of players, the payoff to player i depends only on the total number of players j player $x_j = 1$, i.e. $u(x_i, n_{\neg i}, \beta)$ where $n_{\neg i} := \sum_{j\neq i} x_j$. Note that $n_{\neg i} \in \{0, 1, ..., N-1\}$.

Player i's gain from playing $x_i = 1$ instead of $x_i = 0$, given β and $n_{\neg i}$, is:

$$G(n_{\neg i}, \beta) = u(x_i = 1, n_{\neg i}, \beta) - u(x_i = 0, n_{\neg i}, \beta). \tag{2.2}$$

We make the following assumptions:

(A1) The function G is strictly increasing in β, for all $n_{\neg i}$, for all i.

(A2) The function G is increasing in $n_{\neg i}$, for all β, for all i.

(A3) There exist points $\beta_0 > -\infty$ and $\beta_1 < \infty$ such that, for all i, $G(N-1, \beta_0) = 0$ and $G(0, \beta_1) = 0$. Misschien moet ik deze aanname een tikkeletje anders en pas na de ruis op β geven. Zie mijn mondelinge discussie in sectie 4.2.

Assumption (A2) implies that players' actions are strategic complements.[4] Strategic complementarities cut to the core of equilibrium multiplicity in coordination games where β is common knowledge (Cooper and John, 1988; Van Huyck et al., 1990). By maintaining assumption (A2), we turn the odds of finding a unique equilibrium against ourselves. Assumption (A3) implies the existence of a $\tilde{\beta} \in (\beta_0, \beta_1)$ such that $G(\boldsymbol{x}^M_{\neg i}, \boldsymbol{x}^m_{\neg i}, \tilde{\beta}) = 0$.

[3] These assumptions are restrictive but simplify the analysis substantially. See Morris and Shin (1998, 2002); Angeletos and Pavan (2004); Angeletos and Werning (2006); Angeletos et al. (2007) for examples of other papers that operate under these same assumptions.

[4] For an interesting recent developments of (simultaneous move) global games with strategic substitutes, see Harrison and Jara-Moroni (2020).

While payoffs depend on β, this parameter is unobserved and drawn from a normal distribution $\mathcal{N}(\bar{\beta}, \sigma_\beta^2)$. Each player i receives a private noisy signal b_i of β, such that:

$$b_i = \beta + \varepsilon_i, \tag{2.3}$$

where ε_i is a noise term drawn i.i.d. from a normal distribution $\mathcal{N}(0, \sigma_\varepsilon^2)$. We write $F(\beta, \boldsymbol{b}_{\neg i} \mid b_i)$ for the conditional posterior distribution of $(\beta, \boldsymbol{b}_{\neg i})$, given b_i.

Lemma 1. *Given b_i, the vector $(\beta, \boldsymbol{b}_{\neg i}) \sim \mathcal{N}_n((1-\lambda)\bar{\beta} + \lambda b_i), \Sigma')$, where $\lambda := \sigma_\beta^2/(\sigma_\varepsilon^2 + \sigma_\beta^2)$.*[5] *Importantly,*

(i) The mean vector of $(\beta, \boldsymbol{b}_{\neg i})$ shifts linearly with b_i;

(ii) The covariance matrix Σ' is independent of b_i.

Proof. These are standard properties of the (multivariate) normal distribution. See for example Tong (1990).

Note that $\sigma_\varepsilon \to 0$ implies $\lambda \to 1$ and so $(1-\lambda)\bar{\beta} + \lambda b_i \to (b_i, b_i, ..., b_i)$, that is, when signals become arbitrarily precise players expect both the real β as well as the signals received by all others to be Normally distributed about their own.

3 Simultaneous Moves as a Benchmark

To serve as a benchmark for eventual comparison, this section briefly discuss a simultaneous move global game. In Section 5, we transform the simultaneous move game into a sequential game by partitioning the player set into subsets, leaving the remaining structure of the game as is.

The structure of a simultaneous move global game is common knowledge and as follows:

1. Nature draws a true β.
2. Each $i \in \mathbb{P}$ receives private signal $b_i = \beta + \varepsilon_i$ of β.

[5] The covariance matrix Σ' essentially has two parts. First, a smaller $(n-1) \times (n-1)$ covariance matrix Σ'' for the signals whose diagonal elements are $(\sigma^4 - \sigma_\beta^4)/\sigma^2$ and whose off-diagonal elements $(\sigma^2\sigma_\beta^2 - \sigma_\beta^4)/\sigma^2$. Second, a vector $(\sigma_\varepsilon^2 + \sigma_\beta^2, ...)$ for the variance of β and its covariance with all signals b_j for $j \neq i$, which is the first column/row of Σ'.

3. All $i \in \mathbb{P}$ simultaneously play action $x_i \in \{0, 1\}$.

4. Payoffs are realized according to β and the actions chosen by all players.

In the simultaneous move game, all players choose their actions simultaneously. Since this eliminates the possibility that the action profile $\boldsymbol{x}_{\neg i}$ depends on the realized x_i, we write $G(\boldsymbol{x}_{\neg i}, \beta)$ for the conditional gain of player i.

In terms of notation, for any two vectors $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$, we write $\boldsymbol{y} = \boldsymbol{z}$ if $y_i = z_i$ for all $i = 1, 2, ..., n$. We write $\boldsymbol{y} \geq \boldsymbol{z}$ if $y_i \geq z_i$ for all $i = 1, 2, ..., n$ and $\boldsymbol{y} \neq \boldsymbol{z}$. Finally, we write $\boldsymbol{y} \geqq \boldsymbol{z}$ if $\boldsymbol{y} \geq \boldsymbol{z}$ or $\boldsymbol{y} = \boldsymbol{z}$.

3.1 Iterated Dominance

A pure strategy for player $i \in \mathbb{P}$ is a mapping $s_i : \mathbb{R} \to \{0, 1\}$ projecting signals onto actions. Let $\boldsymbol{b} \in \mathbb{R}^n$ denote a vector $(b_i)_{i \in \mathbb{P}}$ of signals b_i for all agents i, and let $\boldsymbol{b}_{\neg i}$ be a vector of signals for all players but i. We write S_i for the set of strategies for player i, and $S_{\neg i}$ for the set of strategy profiles for all players $j \in \mathbb{P} \setminus \{i\}$, with typical element $\boldsymbol{s}_{\neg i}$.

For $c \in \mathbb{R}$, let $\tilde{c}$ be the function defined by $\tilde{c}(b) = 0$ if $b < c$ and $\tilde{c}(b) = 1$ if $b > c$. If $s_i = \tilde{c}$, we say that s_i is a monotone strategy with switching point c. Everything else constant, the expected gain to playing $x_i = 1$, rather than $x_i = 0$, is increasing in b_i, so it is natural to look at monotone strategies. The monotone strategy *profile* in which all players have switching point c is denoted $\tilde{\boldsymbol{c}}$.

Let player i's expected gain, conditional on b_i and the strategies $s_{\neg i}$ played by all other players, for given σ_ε, be denoted:

$$g^{\sigma_\varepsilon}(\boldsymbol{s}_{\neg i}, b_i) := \iint G(\boldsymbol{s}_{\neg i}(\boldsymbol{b}_{\neg i}), \beta)\, dF(\beta, \boldsymbol{b}_{\neg i} \mid b_i). \tag{3.1}$$

Conditional on the signal b_i, the action $x_i = 1$ is *dominated* if $g_i^{\sigma_\varepsilon}(\boldsymbol{s}_{\neg i}, b_i) < 0$ for all $\boldsymbol{s}_{\neg i}$. Similarly, the action $x_i = 0$ is conditionally dominated if $g_i^{\sigma_\varepsilon}(\boldsymbol{s}_{\neg i}, b_i) > 0$ for all $\boldsymbol{s}_{\neg i}$. Now consider the set of strategies for player i, S_i. Some strategies in the set will be dominated (for example, (A3) implies it is strictly dominated to play $x_i = 0$ for all $b_i > \beta_1$). Stripping the set S_i of all dominated strategies, we obtain a smaller set $S_i^0 \subseteq S_i$, and this is true for each player i. But we know that no player i will ever play a dominated strategy and attention may thus be restricted to the strategies in S_i^0. Moreover, a given player i knows that any $s_{\neg i} \notin S_{\neg i}^0$ is dominated for at least one player

$j \neq i$. Hence, player i should expect his opponents to play a strategy $s_{\neg i}$ from $S^0_{\neg i}$ only. The restriction to action profiles in $S^0_{\neg i}$, however, may lead to additional strategies becoming dominated for player i. Taking those out, player i's set of undominated strategies becomes $S^1_i \subseteq S^0_i$. Again, this is true for all players and player i knows that his opponents will only play strategy profiles belonging to $S^1_{\neg i}$. Yet other strategies for player i may then become dominated, and so on.

The above can be repeated indefinitely and is called iterated elimination of dominated strategies. Iterated dominance is the solution concept of our game. As illustrated in Appendix A, monotone strategies are useful in iteratively defining sets of undominated strategies. This motivates the following Lemma.

Lemma 2. *For all $i \in \mathbb{P}$, for given $c \in \mathbb{R}$ and associated monotone strategy profile $\tilde{\boldsymbol{c}}$,*

$$g^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, b_i) > g^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, b'_i) \iff b_i > b'_i. \tag{3.2}$$

Moreover, for any two $c, d \in \mathbb{R}$ and associated monotone strategy profiles $\tilde{\boldsymbol{c}}_{\neg i}$ and $\tilde{\boldsymbol{d}}_{\neg i}$, for all b_i,

$$g^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, b_i) > g^{\sigma_\varepsilon}(\tilde{\boldsymbol{d}}_{\neg i}, b_i) \iff d > c. \tag{3.3}$$

Proof. Follows immediately from Lemma 1 in combination with assumptions (A1) and (A2).

If all players who are not i play a given monotone strategy, then the expected gain to player i is higher if at least one player $j \neq i$ switches to playing $x_j = 1$ at a lower threshold.

Do there exist equilibria in monotone strategies that survive iterated dominance?

Proposition 1. *There exist points $\beta^{**} \leq \beta^*$ such that the monotone strategy profiles $\tilde{\boldsymbol{\beta}}^{**}$ and $\tilde{\boldsymbol{\beta}}^*$ are Bayesian Nash equilibria surviving iterated elimination of strictly dominated strategies. The points β^{**} and β^* are the solution to:*

$$g^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}^*_{\neg i}, \beta^*_1) = g^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}^*_{\neg i}, \beta^*_0) = 0, \tag{3.4}$$

for all $i \in \mathbb{P}$.

The proof of Proposition 1 requires a rather lengthy construction. It is therefore relegated to Appendix A.

By Proposition 1, there always exists at least one equilibrium in monotone strategies. Actually, if $\beta^{**} < \beta^*$ we can even find *two*. There possibly are many more, perhaps in non-monotone strategies as well. Iterated dominance only prescribes behavior when $b_i < \beta^{**}$ or $b_i > \beta^*$.

On the other hand, if $\beta^{**} = \beta^*$, the equilibrium is clearly unique. When do they?

Lemma 3. *Let σ_ε be sufficiently small. For any two points $c, d \in \mathbb{R}$, let $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{d}}$ denote the associated monotone strategy profiles. For all $i \in \mathbb{P}$,*

$$g^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, c) > g^{\sigma_\varepsilon}(\tilde{\boldsymbol{d}}_{\neg i}, d) \iff c > d. \tag{3.5}$$

Proof. Let $\sigma_\varepsilon^2 \to 0$. Then player i's conditional distribution on $\boldsymbol{b}_{\neg i}$ is multivariate normal with mean vector $(b_i, b_i, ..., b_i)$. Hence, conditional on b_i player i believes that $\Pr[b_j > b_i \mid b_i] = \Pr[b_j < b_i \mid b_i] = 1/2$. The conditional distribution of $\boldsymbol{x}_{\neg i}$ is therefore equivalent whether:

(i) All players $j \neq i$ play a monotone strategy with switching point c and player i receives signal $b_i = c$;

(ii) All players $j \neq i$ play a monotone strategy with switching point d and player i receives signal $b_i = d$.

Since, conditional on $\boldsymbol{x}_{\neg i}$, the gain g is strictly increasing in b_i, we conclude that $g^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{\neg i}, c) > g^{\sigma_\varepsilon}(\tilde{\boldsymbol{d}}_{\neg i}, d) \iff c > d$. As this inequality is strict when $\sigma_\varepsilon \to 0$, we can allow $\sigma_\varepsilon > 0$ and the result is still correct. *Q.E.D.*

Proposition 2. *Let σ_ε be sufficiently small. Then the game has a unique equilibrium $\boldsymbol{s}^*$ surviving iterated elimination of strictly dominated strategies. It is the monotone strategy profile $\tilde{\boldsymbol{\beta}}^{\boldsymbol{*}}$.*

Proof. An immediate implication of Lemma 3 and Proposition 1.

For the case of noisy but not too noisy signals, Proposition 2 establishes that the simultaneous move global game has a unique equilibrium. It is a special case of Theorem 1 in Frankel et al. (2003) on simultaneous move global games with strategic complementarities.

4 Two-Stage Sequential Global Games

We now proceed to sequential global games with two stages where players play sequentially according to an ordered partition $\mathscr{P} = \{\{\mathbb{P}_1\}, \{\mathbb{P}_2\}\}$ of the player set $\mathbb{P} = \{1, 2, ..., N\}$, $N \geq 2$. Let $|\mathbb{P}_1| = N_1$ and $|\mathbb{P}_2| = N_2$, so that $N_1 + N_2 = N$. Define $\boldsymbol{x}_1 = (x_i)_{i \in \mathbb{P}_1}$, $\boldsymbol{x}_{1 \setminus i} = (x_j)_{j \in \mathbb{P}_1 \setminus i}$, $\boldsymbol{x}_2 = (x_j)_{j \in \mathbb{P}_2}$, $\boldsymbol{x}_{2 \setminus j} = (x_i)_{i \in \mathbb{P}_2 \setminus j}$. Moreover, a history h at the beginning of stage 2 is denoted $h = \boldsymbol{x}_1$. By the symmetry of our game, the payoff-relevant characteristic of any action profile $\boldsymbol{x}_t$ is the number of x_is in $\boldsymbol{x}_t$ that are 1. Therefore, for $t = 1, 2$, define $n_t = \sum_{i \in \mathbb{P}_t} x_i$.

The timing of the game is as follows.

1. Nature draws a true β.
2. Each $i \in \mathbb{P}$ receives private signal $b_i = \beta + \varepsilon_i$ of β.
3. All $i \in \mathbb{P}_1$ simultaneously play action $x_i \in \{0, 1\}$.
4. All $i \in \mathbb{P}_2$ observe the history $h = \boldsymbol{x}_1$.
5. All $i \in \mathbb{P}_2$ simultaneously play action $x_i \in \{0, 1\}$.
6. Payoffs are realized according to β and the actions chosen by all players.

It is inconsequential that all players receive their signals simultaneously at the start of the game. None of our results would change if, instead, second-stage players receive their signals only at the start of stage 2.

We make the following additional assumption.

(A4') The history h is accurately observed by players in $\mathbb{P}_2$.

4.1 Strategies and gains

A strategy for player $i \in \mathbb{P}_1$ is a function $s_i : \mathbb{R} \to \{0, 1\}$. Once players in the first stage have chosen their actions, the history h is realized, where $h = (x_i)_{i \in \mathbb{P}_1}$ is a vector in $\mathcal{H} = \{0, 1\}^{N_1}$. We write $\boldsymbol{y}_1$ for the profile $(y_i)_{i \in \mathbb{P}_1}$ and $\boldsymbol{y}_{1 \setminus i}$ for the profile $(y_j)_{j \in \mathbb{P}_1 \setminus \{i\}}$.

Each player $j \in \mathbb{P}_2$ observes both its private signal b_i and the history h, so a strategy for $j \in \mathbb{P}_2$ is a function $s_j : \mathbb{R} \times \mathcal{H} \to \{0, 1\}$. For $j \in \mathbb{P}_2$, we say that a strategy is monotone if it is monotone for every history h. We write $\boldsymbol{y}_{2 \setminus j}$ for the profile $(y_l)_{l \in \mathbb{P}_2 \setminus \{j\}}$.

For player $i \in \mathbb{P}_1$, the conditional gain is given by:

$$G_1(\boldsymbol{x}_{1\setminus i}, \boldsymbol{x}_2^h, \beta) = u(x_i = 1, \boldsymbol{x}_{1\setminus i}, \boldsymbol{x}_2^h \mid \beta) - u(x_i = 0, \boldsymbol{x}_{1\setminus i}, \boldsymbol{x}_2^h \mid \beta), \tag{4.1}$$

where the subscript is added to clearly distinguish between first-stage and second-stage cohorts. By definition $h = (\boldsymbol{x}_{1\setminus i}, x_i)$, which is different when $x_i = 1$ and $x_i = 0$. Hence, the gain G_1 in (4.1) can describe the case where second-stage players play different actions depending on player i's action. Note also that $\boldsymbol{x}_{\neg i} = (\boldsymbol{x}_{1\setminus i}, \boldsymbol{x}_2)$, so (4.1) is perfectly consistent with the definition of G given in (2.2). While we could, for full generality, define (4.1) also for differentiated action profiles $\boldsymbol{x}_{1\setminus i}$, first-stage players choose their actions simultaneously so this is unnecessary.

Player $j \in \mathbb{P}_2$, define the gain from playing $x_j = 1$, rather than $x_j = 0$, to be:

$$G_2(h, \boldsymbol{x}_{2\setminus j}, \beta) := u(x_j = 1, h, \boldsymbol{x}_{2\setminus j} \mid \beta) - u(x_j = 0, h, \boldsymbol{x}_{2\setminus j} \mid \beta). \tag{4.2}$$

From Lemma 1, recall that player i's posterior on the signals received by other players is distributed $\boldsymbol{b}_{1\setminus i} \sim \mathcal{N}_{n_1-1}(\lambda\bar{\beta} + (1-\lambda)b_i), \Sigma')$ and $\boldsymbol{b}_2 \sim \mathcal{N}_{n_2}(\lambda\bar{\beta} + (1-\lambda)b_i), \Sigma')$, where Σ' is specified in said Lemma. For player $i \in \mathbb{P}_1$, let $F(\beta, \boldsymbol{b}_{1\setminus i}, \boldsymbol{b}_2 \mid b_i)$ denote the joint posterior distribution on β, $\boldsymbol{b}_{1\setminus i}$, and $\boldsymbol{b}_2$. Conditional on the signal b_i as well as the strategy-vectors $\boldsymbol{s}_{1\setminus i}$ and $\boldsymbol{s}_2$, the expected gain to player $i \in \mathbb{P}_1$ is therefore:

$$g_1^{\sigma_\varepsilon}(\boldsymbol{s}_{1\setminus i}, \boldsymbol{s}_2, b_i) = \iiint G_1(\boldsymbol{s}_{1\setminus i}(\boldsymbol{b}_{1\setminus i}), \boldsymbol{s}_2(\boldsymbol{b}_2, h) \mid \beta, \boldsymbol{b}_{1\setminus i}, \boldsymbol{b}_2)\, dF(\beta, \boldsymbol{b}_{1\setminus i}, \boldsymbol{b}_2 \mid b_i). \tag{4.3}$$

(Note that a strategy for players in the second stage depends on the realized history. We need not explicitly integrate the gain over all possible histories since, given $\boldsymbol{s}_{1\setminus i}$, the distribution of h is implied by the distribution of $\boldsymbol{b}_{1\setminus i}$.)

Denote by $\mathcal{B}_1^h(\boldsymbol{s}_1) = \prod_{i\in\mathbb{P}_1}\{b_i \mid s_i(b_i) = h_i\}$, where $h_i = x_i$, i.e. is the element of the history h belonging to player i. For player $j \in \mathbb{P}_2$, let $F(\beta, \boldsymbol{b}_{2\setminus j} \mid b_j, \mathcal{B}_1^h(\boldsymbol{s}_1))$ denote the joint posterior distribution on β and $\boldsymbol{b}_{2\setminus j}$, given b_j and $\mathcal{B}_1^h$. The expected gain to player $j \in \mathbb{P}_2$ is then given by:

$$g_2^{\sigma_\varepsilon}(h, \boldsymbol{s}_{2\setminus j}, b_j \mid \boldsymbol{s}_1) = \iint G_2(h, \boldsymbol{s}_{2\setminus j}(\boldsymbol{b}_{2\setminus j}) \mid \boldsymbol{b}_{2\setminus j}, \beta)\, dF(\beta, \boldsymbol{b}_{2\setminus i} \mid b_j, \mathcal{B}_1^h(\boldsymbol{s}_1)). \tag{4.4}$$

4.2 Monotone Strategies and Dominance

This section is devoted to monotone strategies.

The first rationale to explicitly discuss monotone strategies is that they provide "boundaries" for the set of undominated strategies. To see what that means, consider a player i in stage 1 whose private signal satisfies $b_i \geq \beta_1$ (A3). By definition of β_1, this means player i expects to strictly gain from playing $x_i = 1$, rather than $x_i = 0$, even in the extreme event that all other players j play $x_j = 0$ (this extreme case yields the lowest possible payoff by virtue of the strategic complementarities in actions, assumption A2). Technically, this means that any strategy s_i such that $s_i(b_i) = 0$ is strictly dominated (when, as assumed, $b_i \geq \beta_1$). Hence, an undominated strategy s_i must satisfy $s_i(b_i) \geq \tilde{\beta}_1(b_i)$, for all b_i. By a similar argument, an undominated strategy has to satisfy that $s_i(b_i) \leq \tilde{\beta}_0(b_i)$, for all b_i.

The second rationale to use focus on monotone strategies is the informational value contained in a history h when players in stage 1 are assumed never to play a dominated strategy. As we saw in the previous paragraph, any strategy profile $\boldsymbol{s}_1 \leq \tilde{\boldsymbol{\beta}}_1$ is strictly dominated for players in stage 1. For players in stage 2, the "highest posterior" on β, given their private signals, is therefore obtained under the assumption that h is the realization of $\tilde{\boldsymbol{\beta}}_1$ (i.e. this strategy allows for the lowest possible signals b_i to be consistent with $x_i = 0$ in stage one).

Wordt vervolgd.

4.3 Posterior Beliefs and Expectations

Knowing that (i) β is drawn according to $\mathcal{N}(\bar{\beta}, \sigma_\beta^2)$ and (ii) each private signal is simply β plus noise drawn from $\mathcal{N}(0, \sigma_\varepsilon^2)$, a second-stage player j constructs a posterior distribution of β on the basis of his signal b_j and the observed history h. Since β is a priori Normally distributed, the conditional distribution of β is also Normal, though with a different mean and variance.

We are especially interested in the case where first-stage players play a monotone strategy profile $\boldsymbol{s}_1 = \tilde{\boldsymbol{c}}_1$, for some given $c \in \mathbb{R}$. Since h is the realization of $\tilde{\boldsymbol{c}}_1$, player j learns that n_1 players $i \in \mathbb{P}_1$ must have observed a signal $b_1 \geq c$ while $N_1 - n_1$ players i received $b_1 \leq c$. The likelihood of observing the signal b_j and the history h under strategy profile $\tilde{\boldsymbol{c}}_1$, as a function of β, is

$$L(\beta) = \left[1 - \Phi\left(\frac{c-\beta}{\sigma_\varepsilon}\right)\right]^{n_1} \cdot \left[\Phi\left(\frac{c-\beta}{\sigma_\varepsilon}\right)\right]^{N_1-n_1} \cdot \phi\left(\frac{b_j-\beta}{\sigma_\varepsilon}\right) \cdot \phi\left(\frac{\beta-\bar{\beta}}{\sigma_\beta}\right). \quad (4.5)$$

The point $\hat{\beta}_j$ that maximizes the likelihood function L gives us the mean of player j's posterior (Normal) distribution on β.[6]

Lemma 4. *For some given $c \in \mathbb{R}$, let players in the first stage play the monotone strategy profile $\tilde{\boldsymbol{c}}_1$. For player $j \in \mathbb{P}_2$, conditional on its signal b_j and the history h, the vector $(\beta, \boldsymbol{b}_{2\setminus j})$ is multivariate Normally distributed, where:*

(i) The mean of $(\beta, \boldsymbol{b}_{2\setminus j})$ is increasing in b_j;

(ii) The mean of $(\beta, \boldsymbol{b}_{2\setminus j})$ is increasing in n_1 for given c;

(iii) The mean of $(\beta, \boldsymbol{b}_{2\setminus j})$ is increasing in c for given n_1.

Moreover, the conditional covariance matrix Σ_2 of $(\beta, \boldsymbol{b}_{2\setminus j})$ is independent of b_j.

Proof. Player $j \in \mathbb{P}_2$ has makes essentially three observations: (i) his own signal b_j, (ii) the number n_1 of players in $\mathbb{P}_1$ who played 1 and therefore must have observed a signal $b_i \geq c$, and (iii) the number of players in $\mathbb{P}_1$ who played 0 (i.e. $N_1 - n_1$) and therefore must have observed a signal $b_i < c$. Given β, the likelihood $L(\beta)$ of observing the above is given by (4.5).

Define $m(x) := \phi(x)/(1-\Phi(x))$. Define $\xi = (c-\beta)/\sigma_\varepsilon$. Taking the natural logarithm of (4.5) and differentiating with respect to β, we can rewrite:

$$\frac{\partial}{\partial \beta} \ln(L(\beta)) = \frac{n_1}{\sigma_\varepsilon} m(\xi) - \frac{N_1 - n_1}{\sigma_\varepsilon} m(-\xi) + \frac{b_j - \beta}{\sigma_\varepsilon^2} - \frac{\beta - \bar{\beta}}{\sigma_\beta^2}.$$

Let $\hat{\beta}_j$ solve $L'(\hat{\beta}_j) = 0$, our ML-estimator of β. Multiply by σ_ε^2 to obtain:

$$n_1 \cdot m(\hat{\xi}_j)\sigma_\varepsilon - (N_1 - n_1) \cdot m(-\hat{\xi}_j)\sigma_\varepsilon + (b_j - \hat{\beta}_j) - \frac{\sigma_\varepsilon^2}{\sigma_\beta^2}(\hat{\beta}_j - \bar{\beta}) = 0,$$

where $\hat{\xi}_j = (c - \hat{\beta}_j)/\sigma_\varepsilon$. We thus obtain:

$$\lambda b_j + (1-\lambda)\bar{\beta} = \hat{\beta}_j - \lambda n_1 \cdot m(\hat{\xi}_j)\sigma_\varepsilon + \lambda(N_1 - n_1) \cdot m(-\hat{\xi}_j)\sigma_\varepsilon.$$

where $\lambda = \sigma_\beta^2/(\sigma_\beta^2 + \sigma_\varepsilon^2)$. We therefore know that $\hat{\beta}_j$ is (i) increasing in b_j, (ii) increasing in n_1, and (iii) increasing in c. *Q.E.D.*

For given n_1, let $c^{n_1} \in \mathbb{R}$ be a real number so that $\tilde{c}^{n_1}$ is the strategy defined by $\tilde{c}^{n_1}(b) = 0$ for $b < c^{n_1}$ and $\tilde{c}^{n_1}(b) = 1$ for $b > c^{n_1}$, given n_1 (recall that players in stage 2

[6] The Maximum Likelihood estimator for the mean of a Normal distribution is unbiased.

observe the history h or n_1 and can condition their strategies on it). Thus, for a player in the second stage, $\tilde{c}^{n_1}$ denotes the monotone strategy with switching point c^{n_1} when the history of play is n_1.

The following two lemmas, while tedious in terms of notation, formalize several intuitive results. Lemma 5 discusses properties of the expected gain to a first-stage player i. Part (i) says that i's expected gain is decreasing in second-stagers' thresholds, given they play a monotone strategies. Part (ii) says that player i's expected gain is decreasing in fellow fist-stage players' thresholds, given they play monotone strategies. Parts (i) and (ii) are essentially probabilistic statements: given player i's private signal b_i, the probability that any other player j's signal is above some threshold is decreasing in that threshold. Due to the strategic complementarities, i's expected gain is therefore decreasing in the monotone-strategy threshold of his fellow players. Finally, part (iii) says that i's expected gain is increasing in his private signal b_i if all other players play a monotone strategy.

Lemma 5. *(i) For all $i \in \mathbb{P}_1$, for any $c^{n_1}, d^{n_1} \in \mathbb{R}$, all $n_1 = 0, 1, ..., N_1$ and such that $c^{n_1'} \geq c^{n_1}$ and $d^{n_1'} \geq d^{n_1}$ iff $n_1' \geq n_1$, with associated second-stage monotone strategy profiles $\tilde{\boldsymbol{c}}_2$ and $\tilde{\boldsymbol{d}}_2$, for any $\boldsymbol{s}_{1\setminus i}$, for all b_i,*

$$d^{n_1} > c^{n_1} \implies g_1^{\sigma_\varepsilon}(\boldsymbol{s}_{1\setminus i}, \tilde{\boldsymbol{c}}_2^{n_1}, b_i) > g_1^{\sigma_\varepsilon}(\boldsymbol{s}_{1\setminus i}, \tilde{\boldsymbol{d}}_2^{n_1}, b_i). \tag{4.6}$$

(ii) For all $i \in \mathbb{P}_1$, for given $c^{n_1} \in \mathbb{R}$, all $n_1 = 1, 2, ..., N_1$, and associated second-stage monotone strategy profile $\tilde{\boldsymbol{c}}_2^{n_1}$, for given $d, e \in \mathbb{R}$ and associated first-stage monotone strategy profiles $\tilde{\boldsymbol{d}}_{1\setminus i}$ and $\tilde{\boldsymbol{e}}_{1\setminus i}$,

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{d}}_{1\setminus i}, \tilde{\boldsymbol{c}}_2^{n_1}, b_i) > g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{e}}_{1\setminus i}, \tilde{\boldsymbol{c}}_2^{n_1}, b_i) \iff e > d. \tag{4.7}$$

(iii) For all $i \in \mathbb{P}_1$, given $c \in \mathbb{R}$ and the associated first-stage monotone strategy profile $\tilde{\boldsymbol{c}}_{1\setminus i}$, given $d^{n_1} \in \mathbb{R}$, all $n_1 = 0, 1, ..., N_1$, and the associated second-stage monotone strategy profile $\tilde{\boldsymbol{d}}_2^{n_1}$,

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{1\setminus i}, \tilde{\boldsymbol{d}}_2^{n_1}, b_i) > g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{1\setminus i}, \tilde{\boldsymbol{d}}_2^{n_1}, b_i') \iff b_i > b_i'. \tag{4.8}$$

Proof. Parts (i)-(iii) are all based on the fact that, by Lemma 4, player i's conditional probability that $b_j > \bar{b}_j$ is increasing in b_i and decreasing in $\bar{b}_j$. *Q.E.D.*

Lemma 6 describes properties of a second-stage player j's expected gain. Part

(i) says that j's expected gain is increasing in the monotone-strategy threshold of first-stage players, given the remaining second-stage players play a monotone strategy. The reason is that a higher first-stage threshold leads, for given history h, to a higher posterior on β (and signals $\boldsymbol{b}_{2\setminus j}$). Part (ii) says that j's expected gain is decreasing in the monotone-strategy threshold of the other stage 2 players. Part (iii) says that j's expected gain is increasing in his private signal b_j, given all other players play a monotone strategy. And part (iv) says that j's expected gain is higher for histories in which more stage 1 players played 1.

Lemma 6. *(i) For all $j \in \mathbb{P}_2$, for all n_1, given $c^{n_1} \in \mathbb{R}$, all $n_1 = 0, 1, ..., N_1$, and associated second-stage monotone strategy profile $\tilde{\boldsymbol{c}}_2^{n_1}$, for any two $d, e \in \mathbb{R}$ and associated first-stage monotone strategy profiles $\tilde{\boldsymbol{d}}_1$ and $\tilde{\boldsymbol{e}}_1$, for all b_j*

$$g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{c}}_{2\setminus j}^{n_1}, b_j \mid \tilde{\boldsymbol{d}}_1) > g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{c}}_{2\setminus j}^{n_1}, b_j \mid \tilde{\boldsymbol{e}}_1) \iff d > e. \tag{4.9}$$

(ii) For all $j \in \mathbb{P}_2$, for all n_1, for given $c \in \mathbb{R}$ and associated first-stage monotone strategy profile $\tilde{\boldsymbol{c}}_1$, for any $d^{n_1}, e^{n_1} \in \mathbb{R}$, all $n_1 = 0, 1, ..., N_1$, and associated second-stage monotone strategy profiles $\tilde{\boldsymbol{d}}_{2\setminus j}^{n_1}$ and $\tilde{\boldsymbol{e}}_{2\setminus j}^{n_1}$, for all b_j,

$$e_{2\setminus j}^{n_1} > d_{2\setminus j}^{n_1} \implies g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{d}}_{2\setminus j}^{n_1}, b_j \mid \tilde{\boldsymbol{c}}_1) > g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{e}}_{2\setminus j}^{n_1}, b_j \mid \tilde{\boldsymbol{c}}_1). \tag{4.10}$$

(iii) For all $j \in \mathbb{P}_2$, for any $c \in \mathbb{R}$ and associated first-stage monotone strategy profile $\tilde{\boldsymbol{c}}_1$, for any $d^{n_1} \in \mathbb{R}$ and associated second-stage monotone strategy profile $\tilde{\boldsymbol{d}}_{2\setminus j}^{n_1}$, for all n_1,

$$g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{d}}_{2\setminus j}^{n_1}, b_j \mid \tilde{\boldsymbol{c}}_1) > g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{d}}_{2\setminus j}^{n_1}, b_j' \mid \tilde{\boldsymbol{c}}_1) \iff b_j > b_j'. \tag{4.11}$$

(iv) For all $j \in \mathbb{P}_2$, for any $c \in \mathbb{R}$ and associated first-stage monotone strategy profile $\tilde{\boldsymbol{c}}_1$, for any $d^{n_1} \in \mathbb{R}$, all $n_1 = 0, 1, ..., N_1$, and associated second-stage monotone strategy profile $\tilde{\boldsymbol{d}}_{2\setminus j}^{n_1}$, for all b_j,

$$g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{d}}_2^{n_1}, b_j \mid \tilde{\boldsymbol{c}}) > g_2^{\sigma_\varepsilon}(n_1', \tilde{\boldsymbol{d}}_2^{n_1}, b_j \mid \tilde{\boldsymbol{c}}) \iff n_1 \geq n_1'. \tag{4.12}$$

Proof. See Appendix.

An important implication of Lemma 6 is related to what may be called "indifference points". Let players in stage 1 play a monotone strategy profile with threshold vector $\bar{\boldsymbol{b}}_1$. Let all player in stage 2 but j also play a monotone strategy profile with threshold vector $\bar{\boldsymbol{b}}_2$. Then the expected gain to player j is increasing in the history h, as we saw. This means that the signal at which player j is indifferent between playing $x_j = 0$ and $x_j = 1$ is decreasing in h.

4.4 Equilibrium

We solve the game by iterated dominance, backward induction, and forward induction.

For all $i \in \mathbb{P}_1$, we know there exists a point $\bar{\beta}_1^0$ such that it solves:

$$g_1^{\sigma_\varepsilon}(\boldsymbol{x}_{1\setminus i}^m, \boldsymbol{x}_2^m, \bar{\beta}_1^0) = 0. \tag{4.13}$$

Even if every other player j always plays $x_j = 0$, it is beneficial for player i to play $x_i = 1$, rather than $x_i = 0$, for all $b_i > \bar{\beta}_1^0$. This is true by our Assumption (A3). Since the argument applies to all players $i \in \mathbb{P}_1$, we obtain the vector $\bar{\boldsymbol{\beta}}_1^0 := (\bar{\beta}_1^0)$. Assuming that all second-stage players always play 0, each first-stage player i then knows that every player $j \in \mathbb{P}_1 \setminus \{i\}$ plays $x_j = 1$ for all $b_j > \bar{\beta}_1^0$, where $\bar{\beta}_1^0 < \infty$. With that in mind, player i constructs a new threshold $\bar{\beta}_1^1$, and so on. Then, keeping the behavior of players in the second stage by assumption at $\boldsymbol{x}_2^m$, there exists a unique vector β_1^0 such that it solves:

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}_{1\setminus i}^0, \boldsymbol{x}_2^m, \beta_1^0) = 0. \tag{4.14}$$

(Note that β_1^0 is the limit of iterated dominance within the first stage. Existence is an immediate corollary of Proposition 1). Hence, if second-stage players play according to $\boldsymbol{t}_2(\cdot;\infty)$, then the unique best response of players in the first stage is $\boldsymbol{t}_1(\cdot;\boldsymbol{\beta}_1^0)$.

Players in the second stage can figure this out as well. Hence, each $j \in \mathbb{P}_2$, knowing $\boldsymbol{\beta}_1^0$ and assuming all other players $j \in \mathbb{P}_2$ still play $x_j = 0$ always, upon observing the history h, finds a point $\bar{\beta}_2^{h,0}$ that solves:

$$g_2^{\sigma_\varepsilon}(n_1, \boldsymbol{x}_{2\setminus j}^m, \bar{\beta}_2^{n_1,0} \mid \tilde{\boldsymbol{\beta}}_1^0) = 0. \tag{4.15}$$

Again invoking Proposition 1, we find the vector $\boldsymbol{\beta}_2^{h,0}$, which is the unique best response of players in stage 2 to n_1 given $\tilde{\boldsymbol{\beta}}_1^0$:

$$g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{\beta}}_2^{n_1,0}, \beta_2^{n_1,0} \mid \tilde{\boldsymbol{\beta}}_1^0) = 0, \tag{4.16}$$

for each $j \in \mathbb{P}_2$.

If players in the second stage do not play $\boldsymbol{x}_2^m$, however, then the strategy $\boldsymbol{\beta}_1^0$ is no longer iteratively dominant for players in the first stage. Hence, back to the stage one. We define $\boldsymbol{\beta}_1^{k+1}$ as the vector that solves:

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}_{1\backslash i}^k, \tilde{\boldsymbol{\beta}}_2^{h,k+1}, \beta_1^{k+1}) = 0, \tag{4.17}$$

for all $i \in \mathbb{P}_1$ and where $\beta_1^{k+1} \in \boldsymbol{\beta}_1^{k+1}$. For all $h \in \mathcal{H}$, we then define $\boldsymbol{\beta}_2^{h,k+1}$ as the solution to:

$$g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{\beta}}_{2\backslash j}^{h,k}, \beta_2^{h,k+1} \mid \tilde{\boldsymbol{\beta}}_1^{k+1}) = 0, \tag{4.18}$$

for all $j \in \mathbb{P}_2$ and where $\beta_2^{h,k+1} \in \boldsymbol{\beta}_2^{h,k+1}$. Two observations: (i) Each element (for every h) of the vector $\boldsymbol{\beta}_2^{h,k}$ is *increasing* in $\boldsymbol{\beta}_1^k$ and (ii) $\boldsymbol{\beta}_2^{h,k} > \boldsymbol{\beta}_2^{h',k} \iff h \geq h'$. Both are true by virtue of Lemma 6. Moreover, by Lemma 5, we known that if $\boldsymbol{\beta}_1^{k+1}$ solves (4.17), then $\boldsymbol{\beta}_1^{k+1}$ is decreasing in $\boldsymbol{\beta}_2^{h,k+1}$. Hence, we can establish that $|\boldsymbol{\beta}_1^k - \boldsymbol{\beta}_1^{k-m}|$ is decreasing in k, for all $k \geq 1$, all $m \leq k$. Letting $k \to \infty$, the sequence $(\boldsymbol{\beta}_1^k, (\boldsymbol{\beta}_2^{h,k}))$ therefore converges. Call its limit $(\boldsymbol{\beta}_1^*, (\boldsymbol{\beta}_2^{h,*}))$.

We constructed the sequence $(\boldsymbol{\beta}_1^k, (\boldsymbol{\beta}_2^{h,k}))$ by finding points for which an individual player, assuming all other players play 0 unless 1 is dominant is best off playing 1. A similar exercise can be done to find points for which playing 0 is dominant. We obtain the sequence $(\boldsymbol{\beta}_{1,0}^k, (\boldsymbol{\beta}_{2,0}^{h,k}))$, which is also converging to a limit we call $(\boldsymbol{\beta}_1^{**}, (\boldsymbol{\beta}_2^{h**}))$.

Does the sequential global game have equilibria in monotone strategies? From our construction of $(\boldsymbol{\beta}_1^k, (\boldsymbol{\beta}_2^{h,k}))$ and $(\boldsymbol{\beta}_1^k, (\boldsymbol{\beta}_2^{h,k}))$, we can see that the limits are characterized by

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}_{1\backslash i}^{**}, \tilde{\boldsymbol{\beta}}_2^{h**}, \beta_1^{**}) = g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{\beta}}_{2\backslash j}^{h**}, \beta_2^{h**} \mid \tilde{\boldsymbol{\beta}}_1^{**}) = 0, \tag{4.19}$$

and

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}_{1\backslash i}^{*}, \tilde{\boldsymbol{\beta}}_2^{h*}, \beta_1^{*}) = g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{\beta}}_{2\backslash j}^{h*}, \beta_2^{h*} \mid \tilde{\boldsymbol{\beta}}_1^{*}) = 0, \tag{4.20}$$

respectively, for all $i \in \mathbb{P}_1$ and all $j \in \mathbb{P}_2$, for all h. From Lemma 5, we know that $g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{\beta}}_{1\backslash i}^{**}, \tilde{\boldsymbol{\beta}}_2^{h**}, b_i)$ is strictly increasing in b_i. Since $b_i = \beta_1^{**}$ solves the first part of (4.19), the expected gain of player $i \in \mathbb{P}_1$ from playing $x_i = 1$ is negative for all $b_i < \beta_1^{**}$ and positive for all $b_i > \beta_1^{**}$. Hence, the only undominated response to $\boldsymbol{s}_{\neg i} = (\tilde{\boldsymbol{\beta}}_1^{**}, \tilde{\boldsymbol{\beta}}_2^{h**})$ is $\tilde{\beta}_1^{**}$), which is true for all $i \in \mathbb{P}_1$. Similarly, for any player $j \in \mathbb{P}_2$, the expected gain $g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{\beta}}_2^{h**}, b_j \mid \tilde{\boldsymbol{\beta}}_1^{**})$ is strictly increasing in b_j and zero for $b_j = \beta_2^{n_1**}$ (given n_1). The unique undominated response to $\boldsymbol{s}_{\neg j} = (\tilde{\boldsymbol{\beta}}_1^{**}, \tilde{\boldsymbol{\beta}}_2^{h**})$ is therefore $\tilde{\beta}_2^{n_1**}$, for all n_1, for all $j \in \mathbb{P}_2$. It follows that the monotone strategy profile $(\tilde{\boldsymbol{\beta}}_1^{**}, \tilde{\boldsymbol{\beta}}_2^{h**})$ is a perfect Bayesian equilibrium of the sequential global game.

Theorem 1. *For any partition* $\mathscr{P} = \{\{\mathbb{P}_1\}, \{\mathbb{P}_2\}\}$ *of* $\mathbb{P}$*, there exist points* $(\beta_1^{**}, (\beta_2^{n_1**}))$ *and* $(\beta_1^{*}, (\beta_2^{n_1*}))$*, all* $n_1 \in \{0, 1, ..., N_1\}$*, such that the associated monotone strategy profiles* $(\tilde{\boldsymbol{\beta}}_1^{**}, \tilde{\boldsymbol{\beta}}_2^{n_1**})$ *and* $(\tilde{\boldsymbol{\beta}}_1^{*}, \tilde{\boldsymbol{\beta}}_2^{n_1*})$ *are equilibria of the sequential global game. The vectors* $(\beta_1^{**}, (\beta_2^{n_1**}))$ *and* $(\beta_1^{*}, (\beta_2^{n_1*}))$ *solve* (4.19) *and* (4.20)*, respectively.*

Proof. Follows immediately from the preceding construction.

By construction, $\beta_1^{**} \leq \beta_1^{*}$ and $\beta_2^{n_1**} \leq \beta_2^{n_1*}$, for all h. These points demarcate the boundaries for strict dominance of $x = 1$ and $x = 0$, respectively. Hence, if they coincide and $(\boldsymbol{\beta}_{1,0}^{*}, (\boldsymbol{\beta}_{2,0}^{h*})) = (\boldsymbol{\beta}_{1,1}^{*}, (\boldsymbol{\beta}_{2,1}^{h*})) = (\boldsymbol{\beta}_1^{*}, (\boldsymbol{\beta}_2^{h*}))$, the monotone strategy profile with threshold-vector $(\boldsymbol{\beta}_1^{*}, (\boldsymbol{\beta}_2^{h*}))$ is the *unique* equilibrium surviving iterated dominance.

When do $(\boldsymbol{\beta}_1^{**}, (\boldsymbol{\beta}_2^{h**}))$ and $(\boldsymbol{\beta}_1^{*}, (\boldsymbol{\beta}_2^{h*}))$ coincide?

4.5 Unique Equilibrium

Lemma 7. *Let* $\sigma_\varepsilon \to 0$*. For some real number* $c \in \mathbb{R}$*, let first-stage players play the associated monotone strategy profile* $\tilde{\mathbf{c}}_1$*. Given* n_1 *and* b_j*, the conditional posterior on* $(\beta, \boldsymbol{b}_{2\setminus j})$ *of player* j *is Normally distributed with mean vector* $\hat{\boldsymbol{\beta}}_j = (\hat{\beta}_j, \hat{\beta}_j, ..., \hat{\beta}_j)$ *and* $\hat{\beta}_j$ *given by*

$$\hat{\beta}_j = \begin{cases} \frac{b_j + n_1 \cdot c}{1 + n_1} & \text{if} \quad b_j \geq c \\ \frac{b_j + (N_1 - n_1) \cdot c}{1 + N_1 - n_1} & \text{if} \quad b_j \leq c \end{cases}. \tag{4.21}$$

Importantly, note that $\hat{\beta}_j$ *is linear in* b_j *and* c*. The conditional covariance matrix of* $(\beta, \boldsymbol{b}_{2\setminus j})$ *is independent of both* b_j *and* c*.*

Proof. Recall from the proof of Lemma 4 that the expected value $\hat{\beta}_j$ is implicitly defined by:

$$n_1 \cdot m(\hat{\xi}_j)\sigma_\varepsilon - (N_1 - n_1) \cdot m(-\hat{\xi}_j)\sigma_\varepsilon + (b_j - \hat{\beta}_j) - \frac{\sigma_\varepsilon^2}{\sigma_\beta^2}(\hat{\beta}_j - \bar{\beta}) = 0,$$

where $m(x) = \phi(x)/(1 - \Phi(x))$ and $\xi_j = (c - \hat{\beta}_j)/\sigma_\varepsilon$. Observe that $\sigma_\varepsilon \to 0$ implies $\xi \to \pm\infty$. We know that $m(x)/x \to 1$ if $x \to +\infty$ and $m(x)/x \to 0$ for $x \to -\infty$. Finally, since $\hat{\xi} = (c - \hat{\beta})/\sigma_\varepsilon$, we note that $m(\hat{\xi})\sigma_\varepsilon$ can be rewritten as $(m(\hat{\xi})/\hat{\xi})(c - \hat{\beta})$. Solving for $\hat{\beta}$ yields the result. *Q.E.D.*

For two real number $c, d \in \mathbb{R}$, consider the associated first-stage monotone strategy profiles $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{d}}$. Without loss of generality, let $d \geq c$ and define $\Delta := d - c$. Also, for any $n_1 = 0, 1, ..., N_1$, take two real numbers $y^{n_1}, z^{n_1} \in \mathbb{R}$ and consider the associated second-stage monotone strategy profiles $\tilde{\boldsymbol{y}}_2^{n_1}$ and $\tilde{\boldsymbol{z}}_2^{n_1}$. Then if $\sigma_\varepsilon \to 0$, stage 1 player i's posterior distribution of $\boldsymbol{x}_{\neg i}$ is equivalent whether all other players play strategy profile (i) $\boldsymbol{s}_{\neg i} = (\tilde{\boldsymbol{c}}_{1\setminus i}, \tilde{\boldsymbol{y}}_2^{n_1})$ and $b_i = c$ or (ii) $\boldsymbol{s}_{\neg i} = (\tilde{\boldsymbol{d}}_{1\setminus i}, \tilde{\boldsymbol{z}}_2^{n_1})$ and $b_i = d$ if and only if $z^{n_1} = y^{n_1} + \Delta$ for all n_1. (This is an immediate implication of Lemma 1).

Having established the above, for any player i in stage 1, for all $\Delta > 0$ it follows,

$$g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{c}}_{1\setminus}, \tilde{\boldsymbol{y}}_2^{n_1}, c) > g_1^{\sigma_\varepsilon}(\tilde{\boldsymbol{d}}_{1\setminus}, \tilde{\boldsymbol{z}}_2^{n_1}, d), \tag{4.22}$$

if $z^{n_1} \geq y^{n_1} + \Delta$ for all n_1.

For any n_1 and any $\Delta > 0$, we also know that:

$$g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{z}}_{2\setminus j}^{n_1}, z^{n_1} \mid \tilde{\boldsymbol{d}}_1) > g_2^{\sigma_\varepsilon}(n_1, \tilde{\boldsymbol{y}}_{2\setminus j}^{n_1}, y^{n_1} \mid \tilde{\boldsymbol{c}}_1), \tag{4.23}$$

if $z^{n_1} = y^{n_1} + \Delta$.

Now apply the inequalities in (4.22) and (4.23) to the monotone equilibrium strategy profiles $(\tilde{\boldsymbol{\beta}}_1^{**}, \tilde{\boldsymbol{\beta}}_2^{n_1**})$ and $(\tilde{\boldsymbol{\beta}}_1^{*}, \tilde{\boldsymbol{\beta}}_2^{n_1*})$. By construction, we know that $\beta_1^* \geq \beta_1^{**}$ and $\beta_2^{n_1*} \geq \beta_2^{n_1**}$, for all n_1. Defining $\Delta := \beta_1^* - \beta_1^{**}$, we conclude from (4.22) that $\Delta > 0$ can be an equilibrium for players in stage 1 if and only if $\beta_2^{n_1*} > \beta_2^{n_1**} + \Delta$ for at least one n_1. However, by (4.23) we know that *if* $(\tilde{\boldsymbol{\beta}}_1^{**}, \tilde{\boldsymbol{\beta}}_2^{n_1**})$ is an equilibrium, *then* $(\tilde{\boldsymbol{\beta}}_1^{*}, \tilde{\boldsymbol{\beta}}_2^{n_1*})$ can be an equilibrium for players in stage 2 if and only if $\beta_2^{n_1*} < \beta_2^{n_1**} + \Delta$ for all n_1, contradiction the fact that $\beta_2^{n_1*} \geq \beta_2^{n_1**}$.

Theorem 2. *For σ_ε sufficiently small, for any partition $\mathscr{P} = \{\{\mathbb{P}_1\}, \{\mathbb{P}_2\}\}$ of $\mathbb{P}$, the two-stage sequential global game has a unique equilibrium in monotone strategies that survives iterated dominance, forward induction, and backward induction. In the equilibrium, $\boldsymbol{s}_1 = \tilde{\boldsymbol{\beta}}_1^*$ and $\boldsymbol{s}_2(\cdot, n_1) = \tilde{\boldsymbol{\beta}}_2^{n_1*}$, all $h \in \mathcal{H}$.*

Proof. The result is immediate from applying Lemmas 1 and 7 to the (potentially distinct) monotone equilibria with thresholds $(\boldsymbol{\beta}_1^{**}, \boldsymbol{\beta}_2^{n_1**})$ and $(\boldsymbol{\beta}_1^{*}, \boldsymbol{\beta}_2^{n_1*})$ in Theorem 1 when $\sigma_\varepsilon \to 0$, see also the preceding discussion. *Q.E.D.*

4.6 Efficiency

If $\sigma_\varepsilon \to 0$, there is a unique equilibrium with thresholds $(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^{h*})$. What can we say about the efficiency of this equilibrium?

Lemma 8. *Let σ_ε be sufficiently small. For the vectors $(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^{h*})$ from Theorem 2, it holds*

$$\beta_2^{N_1 *} < \beta_1^* < \beta_2^{0*}, \tag{4.24}$$

where $(1, 1, ..., 1)$ *is the history in which all first-stage players played 1 and* $(0, 0, ..., 0)$ *the history in which all first-stage players played 0.*

By Lemma 8, the outcome of a sequential global with arbitrarily precise signals is completely determined once we know $\beta \lesseqgtr \beta_1^*$. If $\beta > \beta_1^*$, the outcome of the game is $(1, 1, ..., 1)$ with probability 1. This is so because $\beta > \beta_1^*$ implies that $\boldsymbol{x}_1 = (1, 1, ..., 1)$ with probability 1, which, since $\beta > \beta_1^*$ implies $\beta > \beta_2^{(1,1,...,1)*}$, also means that $\boldsymbol{x}_2 = (1, 1, ..., 1)$ with probability 1. Symmetrically, if $\beta < \beta_1^*$, then the outcome of the game is $(0, 0, ..., 0)$ almost surely. It follows that $\beta_1^* < \beta^*$, where β^* is the unique equilibrium threshold in the simultaneous move global game (Proposition 2).

Recall that for $\beta < \beta_0$, the only efficient action profile is $(0, 0, ..., 0)$ whereas for $\beta > \beta_0$, the only efficient action profile is $(1, 1, ..., 1)$. Inefficiency thus arises when $\beta \in (\beta_0, \beta^*)$ in the simultaneous move game and $\beta \in (\beta_0, \beta_1^*)$ in the sequential game. (In those cases, the efficient outcome is $(1, 1, ..., 1)$ whereas player coordinate on $(0, 0, ..., 0)$). Since $\beta_1^* < \beta^*$, it is immediate that

$$\Phi\left(\frac{\beta_1^* - \bar{\beta}}{\sigma_\beta}\right) < \Phi\left(\frac{\beta^* - \bar{\beta}}{\sigma_\beta}\right).$$

The following theorem is now immediate.

Theorem 3. *For σ_ε sufficiently small, the ex ante expected welfare in a sequential global game is higher than in a simultaneous move global game.*

The equilibrium of a sequential global game yields strictly higher welfare, in expectations, than that of a simultaneous move global game. Theorem 3 may therefore have some policy implications, such as when the interactions between individuals who are (in abstracto) playing a global game are to be regulated. In these cases, a benevolent planner might prefer to let individuals act sequentially since this increases the likelihood that an efficient outcome is reached.

Since $\mathbb{P}$ is discrete and finite, observe that n_1 suffices to describe the partition $\mathscr{P} = \{\{\mathbb{P}_1\}, \{\mathbb{P}_2\}\}$ of $\mathbb{P}$, where $n_1 := |\mathbb{P}_1|$. Moreover, we note that n_1 can only take a finite number of values, since $n_1 \in \{1, 2, ..., n-1\}$. Hence, the set of possible partitions (into two subsets) of $\mathbb{P}$ is finite. It follows that there exists a partition n_1^* such that, for any other partition $n_1 \neq n_1^*$, $\beta_1^*(n_1^*) \leq \beta_1^*(n_1)$.

Corollary 1. *Let σ_ε be sufficiently small. For $\mathbb{P} = \{1, 2, ..., n\}$, let $\beta_1^*(n_1)$ be the unique equilibrium threshold for players in the first stage induced by the partition $\mathscr{P}$ with n_1 first-stage players. There exists a n_1^* such that $\beta_1^*(n_1^*) \leq \beta_1^*(n_1)$ for all $n_1 \neq n_1^*$.*

Corollary 1 follows trivially from Theorem 2 and motivates an interesting question: what is the relationship between $\beta_1^*(n_1)$ and n_1, for given n? How does the decomposition of a player set $\mathbb{P}$ into sequentially playing cohorts affect the expected efficiency of a sequential global game?

5 T-stage Sequential Global Games

In this section, we present a condensed analysis of sequential global games where the player set $\mathbb{P}$ is partitioned into $T \geq 2$ stages. Since the results given here are generalizations of those for the 2-stage sequential global game discussed in the previous section, constructions and proof are relegated to the Appendix.

Let a partition $\mathscr{P} = \{\{\mathbb{P}_1\}, \{\mathbb{P}_2\}, ..., \{\mathbb{P}_T\}\}$ of $\mathbb{P}$ split the player set up into *cohorts* of players $\mathbb{P}_t$, all $t = 1, 2, ..., T$, where t is referred to as a *stage*. At the start of stage t, the history h_t is the vector of actions played by all players in all preceding stages, i.e. $h_t = (\boldsymbol{x}_s)_{s=1}^{t-1}$, and $h_1 = \emptyset$. Each player i chooses an action $x_i \in \{0, 1\}$. We consider a game where all private signals b_i of β are drawn and observed before players choose their actions; the game where players in any stage t receive their signals only at the start of stage t can be analyzed in exactly the same way.

The timing of the game is as follows.

1. Nature draws a true β.

2. Each $i \in \mathbb{P}$ receives private signal $b_i = \beta + \varepsilon_i$ of β.

3. All $i \in \mathbb{P}_1$ simultaneously play action $x_i \in \{0, 1\}$.

4. All $i \in \mathbb{P}_2$ observe the history $h_2 = \boldsymbol{x}_1$.

5. All $i \in \mathbb{P}_2$ simultaneously play action $x_i \in \{0,1\}$.

   $\vdots$

6. All $i \in \mathbb{P}_T$ observe the history $h_T = (\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{T-1})$.

7. All $i \in \mathbb{P}_T$ simultaneously play action $x_i \in \{0,1\}$.

8. Payoffs are realized according to β and the actions chosen by all players.

We maintain the following assumption.

(A4) For all $t = 1, 2, ..., , T$, the history h_t is accurately observed by players in $\mathbb{P}_t$.

A strategy s_i for player i in stage t maps any observed history h_t and signal b_i to an action x_i. A strategy for player i in stage t is *monotone* if, given any history h_t, $s_i(b_i, h_t) = 1 \implies s_i(b_i', h_t) = 1$ for all $b_i' \geq b_i$. For notational purposes, we shall at times write $\boldsymbol{s}_{s \lesseqgtr t}$, by which we mean the strategy profile of all players in all stages $s \lesseqgtr t$. Similarly, $\boldsymbol{b}_{s \lesseqgtr t}$ refers to the vector of signals received by all players in all stages $s \lesseqgtr t$.

For a player i in stage $t \in \{1, 2, ..., T\}$, define the gain from playing $x_i = 1$, rather than $x_i = 0$, to be G_t, given by:

$$G_t(\boldsymbol{x}^0_{\neg i}, \boldsymbol{x}^1_{\neg i}, \beta) = u(x_i = 1, h_t, \boldsymbol{x}_{s\setminus i}, \boldsymbol{x}^1_{t>s}, \beta) - u(x_i = 0, h_t, \boldsymbol{x}_{s\setminus i}, \boldsymbol{x}^0_{t>s}, \beta). \tag{5.1}$$

Since the action of player i in stage t is observed by players in all stages $s > t$, we need to condition the action profile $\boldsymbol{x}_{s>t}$ on the action played by player i.

Contingent on the history h_t and the strategy profile $\boldsymbol{s}_{s<t}$ (of which h_t is the resultant), given the signal b_i, player i in stage t forms a posterior on β, the signals received by his fellow stage t players, and the signals received by those players in stages $s > t$. Let $F(\beta, \boldsymbol{b}_{t\setminus i}, \boldsymbol{b}_{s>t}; h_t, \boldsymbol{s}_{s<t}, b_i)$ denote the distribution function of player i's posterior. The conditional expected gain for player i in stage t is given by:

$$\begin{aligned} &g_t^{\sigma_\varepsilon}(\boldsymbol{x}_{s<t}, \boldsymbol{s}_{t\setminus i}, \boldsymbol{s}_{s>t}, b_i) = \\ &\iiint G_t(\boldsymbol{x}_{s<t}, \boldsymbol{s}_{t\setminus i}(\boldsymbol{b}_{t\setminus i}, h_t), \boldsymbol{s}_{s>t}(\boldsymbol{b}_{s>t}, h_s), \beta)\, dF(\beta, \boldsymbol{b}_{t\setminus i}, \boldsymbol{b}_{s>t}; h_t, \boldsymbol{s}_{s<t}, b_i). \end{aligned} \tag{5.2}$$

Consider a player i in a stage t. If all other players, in all stages, play monotone strategies, then for any history h_t the expected gain from playing $x_i = 1$, rather than $x_i = 0$, is increasing in the signal b_i. Lemma 10 formalizes this result, which has a clear intuition. If all other players pursue a monotone strategy, the likelihood of them playing 1, rather than 0, is strictly increasing in β as signals are drawn around β. Moreover, player i's posterior on β shifts to the right if b_i goes up. Since i's gain is increasing in both $\boldsymbol{x}_{\neg i}$ and β, the expected gain is increasing in the signal b_i.

For a set of thresholds $(\bar{b}_s^{h_s})$, all $= 1, 2, ..., T$, all h_s, the monotone strategy profile $(\boldsymbol{t}_s(\cdot; \bar{\boldsymbol{b}}_s^{h_s})_{s=1}^T)$ is an equilibrium of the T-stage sequential global game if and only if, for all $i \in \mathbb{P}_t$, for all $t = 1, 2, ..., T$, it holds:

$$g_t^{\sigma_\varepsilon}(h_t(\boldsymbol{t}_{s<t}(\cdot; \bar{\boldsymbol{b}}_s^{h_s})), \boldsymbol{t}_{t \setminus i}(\cdot; \bar{\boldsymbol{b}}_t^{h_t}), \boldsymbol{t}_{q>t}(\cdot; \bar{\boldsymbol{b}}_q^{h_q}), \bar{b}_t^{h_t}) = 0. \tag{5.3}$$

The monotone strategy profile $(\boldsymbol{t}_s(\cdot; \bar{\boldsymbol{b}}_s^{h_s})_{s=1}^T)$ is an equilibrium since, for each player i in any period t, a unilateral deviation from this profile yields a strictly lower expected gain/payoff. To see this, note that any other strategy s_i will either prescribe $x_i = 1$ for some $b_i < \bar{b}_t^{h_t}$ or $x_i = 0$ for some $b_i > \bar{b}_t^{h_t}$. But these actions are conditionally dominated since player i's expected gain is increasing in b_i when the other players play monotone strategies and $b_i = b_t^{h_t}$ solves (5.3). Therefore player i is worse off unilaterally deviating from $(\boldsymbol{t}_s(\cdot; \bar{\boldsymbol{b}}_s^{h_s})_{s=1}^T)$, establishing the profile is an equilibrium indeed.

Does a T-stage sequential global game have equilibria in monotone strategies?

Theorem 4. *The monotone strategy profiles $(\boldsymbol{t}_s(\cdot; \boldsymbol{\beta}_{s,1}^{h_s *}))_{s=1}^T$ and $(\boldsymbol{t}_s(\cdot; \boldsymbol{\beta}_{s,0}^{h_s *}))_{s=1}^T$ are perfect Bayesian equilibria of the game.*

Proof. See Appendix.

When the noise in signals becomes sufficiently small, the threshold vectors $(\boldsymbol{\beta}_{0,s}^{h_s *})$ and $(\boldsymbol{\beta}_{1,s}^{h_s *})$ coincide. In this case, iterated elimination of dominated strategies forces players in a T-stage sequential global game to coordinate on a unique equilibrium.

Theorem 5. *For σ_ε sufficiently small, for any partition $\mathscr{P}$ of $\mathbb{P}$, a sequential global game has a unique equilibrium surviving iterated elimination of dominated strategies. In the equilibrium, $\boldsymbol{s}^* = (\boldsymbol{t}_s(\cdot; \boldsymbol{\beta}_s^{h_s *}))_{s=1}^T$, where $(\boldsymbol{\beta}_s^{h_s *}) = (\boldsymbol{\beta}_{0,s}^{h_s *}) = (\boldsymbol{\beta}_{1,s}^{h_s *})$.*

Proof. See Appendix.

6 Applications

The preceding analyses were fairly abstract. We now present several applications that fit well into the theoretical framework of a sequential global game.

6.1 Network Goods

The utility a consumer derives from consuming a network good is increasing with the number of other individuals consuming it (Katz and Shapiro, 1985). One could think of telephones, the Internet, social media, certain types of software, and club membership as some, among many more, examples.

Let there be a pool of potential consumers each of which decides whether or not to buy some network good against a price p. We write $x_i = 1$ if consumer i buys the good and $x_i = 0$ if i does not buy it. (The model can equivalently be interpreted as one where consumers have to choose between two goods. The action $x_i = 1$ then corresponds to buying one good whereas $x_i = 0$ corresponds to buying the other.) Given individual consumers' choices, the good's network size is denoted X:

$$X = \sum_{i \in \mathbb{P}} x_i. \tag{6.1}$$

If the good is a network good, an individual consumer's payoff to buying the good is strictly increasing in X. For simplicity, we let this 'network externality' be linear.

The good's *quality* β is drawn from a Normal distribution with mean $\bar{\beta}$ and variance σ_β^2. We assume that a higher quality β increases consumers' consumption-utility or willingness to pay. Conditional on X, the payoff to consumer i is then:

$$u_i = (\beta + X - p)x_i, \tag{6.2}$$

which we normalized to 0 for $x_i = 0$, so payoffs u_i coincide with our gain function G.

If we assume that β, once drawn, is common knowledge, our model is similar to Katz and Shapiro's (1985) and can have multiple equilibria. For example, let there be four consumers, let $\beta = 0$, and suppose $p = 3.5$. If, as in Katz and Shapiro, all consumers take decisions simultaneously, the model has two Nash equilibria: all buy the good and none buy it. Similarly, the sequential game where two consumers choose whether to buy the good in the first stage while the remaining two consumers decide in the second has two subgame perfect equilibria: all buy, or none do.

Our conclusion that the classic model has multiple (and wildly different) equilibria is somewhat unsatisfying. Economic intuition would suggest that a commodity's quality should play a role in determining its eventual fate. Katz and Shapiro appear to share this view. "*Given the possibility of multiple equilibria*," they write, "[...] *firms' reputations* *may play a major role in determining which equilibrium actually obtains. For example, the existence of a strong reputation for being a market share leader may explain IBM's rapid rise to preeminence in the personal computer market.*"

We argue that the standard model's lack of predictive power is an artifact of its not incorporating two features that characterize the market for network goods. These are (i) a good's quality is often uncertain, especially when the good is new; and (ii) individuals do not all at once decide whether or not to buy the good.

The model described thus far may be adapted to incorporate feature (i) as follows: once the real quality β is drawn, consumers do not actually observe it. Instead, each consumer i receives a private noisy signal b_i of β, where $b_i = \beta + \varepsilon_i$ with $\varepsilon_i \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ individual-specific noise. While the payoff to consumer i still depends on β, this parameter is unobserved. The expected payoff to player i (given X) is therefore:

$$u_i^e = \int (\beta + X - p) x_i \, dF(\beta \mid b_i), \tag{6.3}$$

where $F(\beta \mid b_i)$ is the posterior density on β given the signal b_i.

The assumption of noisy signals on an unobserved β turns the model à la Katz and Shapiro into a simultaneous move global game. As we saw in Section 3, this model has a unique equilibrium.

But while the simultaneous move global game has a unique equilibrium, it cannot formalize such notions as "reputations" or "being a market share leader". To describe these phenomena, we need a game where different groups of consumers buy the good at different points in time: feature (ii). Combined with feature (i), these assumptions make for a sequential global game. As we saw in Sections **??** and 5, the model will have a unique equilibrium. For consumers in stage $t > 1$, let the network size realized in all preceding stages be denoted X_t. The equilibrium has the following properties:

(i) Initial consumers will buy the good if they perceive its quality to be sufficiently high: there exists a unique quality requirement b_1^* such that consumer i buy the good if, and only if, $b_i \geq b^*$.

(ii) In each stage $t > 1$, and given the network size established in previous stages X_t, there exists a unique quality-requirement $b_t^*(X_t)$ such that consumer i in stage t will buy the good if, and only if, $b_i \geq b_t^*(X_t)$.

(iii) In each stage t, the minimum quality requirement $b_t^*(X_t)$ is *decreasing* in the extant network size X_t, that is, in the number of consumers having already bought the good.

In the interpretation of our model as a choice between two different goods, the fact that $b_t^*(X_t)$ is decreasing in X_t translates into what Katz ans Shapiro call the effect of market leadership: high sales of one good compared to the other leads to even higher sales later on and help explain a good's "rise to preeminence".

We have seen how the intuitive yet informal selection mechanism suggested by Katz and Shapiro is fully formalized by the theory of sequential global games. Indeed, their argument can be incorporated into our framework almost one-to-one. It appears that the sequential global games approach can be fruitful for the study of network goods.

6.2 Global Disease Eradication

The eradication of contagious diseases requires coordinated efforts among countries. Coordination is needed, partly, because national eradication efforts are strategic complements. Unless hermetically contained, diseases easily spread borders. Consequently, a given country's eradication efforts are more likely to succeed if a greater number of countries takes eradication efforts too, since this (in probability) reduces the pool of countries from which the disease can spread to the given country.[7] Due to the need for coordination, typical models of disease eradication have multiple equilibria (Barrett, 2003). These models therefore cannot explain why some diseases get eradicated while others do not.

Consider a world consisting of N countries. Each country i can take effort to eradicate a disease ($x_i = 1$, e.g. a lockdown), or not ($x_i = 0$, e.g. no lockdown), and the cost of effort is C. If n countries choose to exert effort, the probability that their efforts are successful and the disease gets (nationally) eradicated is $p(n)$. We assume that $p(n+1) > p(n)$ for all $n = 0, 1, ..., N-1$, where $p(0) = 0$ and $p(N) = 1$.

When a country successfully eradicates the disease, this yields benefits β which is Normally distributed with mean $\bar{\beta}$ and variance σ_β^2. Given n countries $j \neq i$ attempt to eradicate the disease ($x_j = 1$), the payoff to country i is:

$$u_i(x_i; \beta, n) = [p(n + x_i) \cdot \beta - C] \cdot x_i, \tag{6.4}$$

[7] Indeed, even if a country has succeeded in eradicating a disease nationally, it may be reimported again from another country, as the case of smallpox in Botswana illustrates (Fenner et al., 1988).

where $p(n+x_i) \cdot \beta$ is the *expected benefit* from exerting effort. In a model with common knowledge of β (Barrett, 2003), there are multiple (symmetric) equilibria when $\beta \in [C, C/p(1)]$: all countries exerting effort is an equilibrium, but no country exerting effort is an equilibrium as well. Note that a social planner would let all countries exert effort whenever $\beta > C$: the equilibrium where no country eradicates the disease is inefficient in those cases.

The assumption that β is common knowledge may be reasonable in some cases, but it is not always. Especially when a disease is new or understudied, there will be uncertainty about β. Rather than knowing β, let each country i have a private assessment b_i of β. Assume that $b_i = \beta + \varepsilon_i$, where ε_i is the error in country i's assessment, distributed Normally with mean 0 and variance σ_ε^2 and i.i.d. between countries. These assumptions turn the model into a global game, where countries maximize expected payoffs:

$$u_i^e = \int [p(n+x_i) \cdot \beta - C] \cdot x_i \, dF(\beta \mid b_i), \tag{6.5}$$

with $F(\beta \mid b_i)$ the posterior distribution of country i in β, given b_i

Casual observation suggests that countries do not always take efforts to eradicate a disease simultaneously, though. For example, the outbreak of a new disease is normally a local phenomenon. It takes some time for a disease originating in one part of the global to become a large-scale plague another. Because of that, it seems reasonable to assume that some countries face the (political) decision to take eradication efforts earlier than others, where the stage at which a country has to make its effort choice can be thought of as the point in time at which it is affected by the disease. If a disease is characterized by this type of spreading pattern, the machinery of sequential global games is called for.

Our theory generates several predictions. In strong contrast to existing models (Barrett, 2003), the "global eradication game" has a unique equilibrium. In this equilibrium, the countries that are initially affected by the disease (stage 1) take eradication efforts if, and only if, their private assessments b_i of the eradication benefit are above some minimum level b_1^*. In later stages $t > 1$, i.e. for those countries that are affected later, there will similarly be a minimum benefit level $b_t^*(n_t)$ such that countries in stage t take eradication efforts if, and only if, their private benefit assessments are at least $b_t^*(n_t)$, where n_t is the total number of countries that took eradication efforts in all stages up until stage t. The minimum benefit assessment required to take effort

$b_t^*(n_t)$ is decreasing in n_t, the number of countries having taken efforts already.

There is a twofold reason countries are more eager to take efforts when a greater number of countries has done so before them. First, there is an effect operating through the strategic complementary of eradication efforts, since the probability that an individual country's efforts pay off is increasing in the number of fellow eradicators. Second, there is a learning effect since β, the true eradication benefit, is unknown. Each country received a private assessment of the eradication benefits, and these assessments are on average correct. Moreover, a country knows that all other countries will (in the unique equilibrium) take eradication efforts if and only if their private assessments are above some endogenous threshold. Hence, if a country observes that more countries have taken effort, it implies that more countries must have had high benefit assessments, which increases the country's own (posterior) benefit estimate and thus makes eradication more beneficial in expectations.

The minimum benefit estimate required to take eradication efforts is decreasing in C, the cost of effort. While this may seem intuitive or even obvious, it cannot be concluded from standard models with multiple equilibria.

6.3 The Global Climate Game

Climate change constitutes one of the greatest challenges facing modern society. Caused by the combustion of fossil fuels, the only way to avoid catastrophic climate change is a drastic reduction in global CO2 emissions. This presents a twofold problem. On the one hand, the transition to an emission-free economy is costly, requiring large-scale investments in green technologies. On the other hand, climate change is driven by the emissions of all countries – if only a few manage to curb their emissions, this is unlikely to halt global warming.[8] There hence is a clear coordination problem embedded in climate change. For a long time, international environmental agreements (such as the Kyoto Protocol or the Paris Agreement) have been advocated as a workable solution to help overcome these problems. However, their success has been relatively limited and no currently effective agreement is sufficiently strong to avoid severe climate change from happening.

We argue that the sequential global games approach may deepen our understanding of the world's performance on global warming and offer potential solutions. Two characteristics of climate change are important to consider in this regard: (i) here

[8]Moreover, Karp (2017) shows how climate investments are dynamic strategic complements.

are many uncertainties about climate change. We mention the cost of climate change, the costs of a green transition, the existence and location of tipping points, and so on; (ii) climate change is a slow process, allowing countries to join an environmental agreement non-simultaneously without strongly detrimental effects.[9] Recognition of the coordination problem embedded in climate change as well as the points raised above yields a sequential "global climate game". We can draw several lessons.

First, climate change, despite its known averse consequences, may be a rational outcome. As we have seen in both the simultaneous move and the sequential global game, players may settle on an equilibrium of which everyone *knows* it is inefficient. This, perhaps, sheds a light on the at first sight paradoxical observation that most countries seem to recognize climate change as a major problem ideally prevented while, at the same time, no international environmental agreement strong enough to prevent climate change exists.

Second, we stand a greater chance in the fight against climate change if countries could join an international environmental agreement sequentially. As we have seen, the range of fundamentals for which a sequential global game has an efficient equilibrium is strictly larger than for a simultaneous move global game. While the model is agnostic about the countries most suited to be first-movers, richer and technologically-advanced countries stand out natural candidates.

The sequential global games approach advocated here is different from other recent studies trying to improve the depth and reach of international environmental agreements, most notably (Harstad, 2012, 2016; Battaglini and Harstad, 2016). Future work combining these two angles would be interesting.

6.4 Regime Change

Games of regime change are coordination games in which agents can attack a status quo which, if successful, causes a discrete jump in payoffs. The strength of the status quo is an unobserved fundamental of the game that determines the minimum proportion of attackers necessary for the status quo to be abandoned. Whether used to study speculative attacks (Morris and Shin, 1998), bank runs (Goldstein and Pauzner, 2005), political protests (Edmond, 2013), or more abstract situations (Angeletos et al., 2007),

[9] This is important since not all applications naturally allow for sequential actions. Currency attacks, for example, are probably best thought of as a one-shot phenomenon, since the central bank would try to intervene after the first "wave of attack", pre-empting the success of later waves. See Morris and Shin (1998) and also the next subsection.

games of regime change are some of the most-encountered applications of global games.[10]

The most realistic dynamic for a game of regime change varies by application. There essentially are three possible timings: (i) there is one point in time at which all individuals simultaneously decide to attack or not (Morris and Shin, 1998; Edmond, 2013); (ii) there is a repeated number of possibilities to attack, and each time all individuals simultaneously to attack or not (Angeletos et al., 2007); (iii) there is a group of individuals who attack first, followed by others who decide if they join the initial attack (this paper). While most regime changes likely combine aspects of these three dynamics, each "pure" timing will be more or less suitable depending on the application. Some, like currency attacks and bank runs, are most likely a simultaneous move event. Some, like regular elections and membership renewal, occur repeatedly for everyone. And some are probably best described as a sequential game.

Consider political protests aimed at overthrowing an incumbent regime. As many real-world experiences illustrate, a typical order of events has a relatively small group of individuals initiate the protests which, in subsequent days, swell as others join in the efforts. After some time, the fate of the regime is decided. If the regime is strong, it sends the army and ends the protests. If it is weak, the protester win and the regime is overthrown.

The benefit of a sequential global games approach to political protests and regime change is its prediction that weak regimes are inherently more susceptible to protests than strong regimes. The likelihood that protests are successful (rather than followed up by severe punishment and oppression) should intuitively depend on the strength of the regime. Since complete information games of regime change do not generally make this prediction (Siqueira, 2003), the sequential global games approach may provide a fruitful tool to analyze and understand regime changes and political protests.

6.5 Financial Markets and Housing

Financial investors buy shares whose price they expect to rise. The price of a share is influenced by a multitude of factors, including (i) other investors' demand for the share; and (ii) the health of a firm, its industry, and the general economy. Point (i) creates strategic complementarity in stock investments, inducing a coordination game among investors chasing the stocks in highest demand. By this observation alone, one could

[10] It should be clear that the game presented in this paper's main analysis can be further specified to yield a game of regime change.

explain such phenomena as bubbles and the excessive volatility of financial markets (Ganguli and Yang, 2009).

Point (ii) lists fundamentals that affect the price of a stock but of which no investor can be fully knowledgeable without error. Investors will have beliefs about them, though. Combining (i) and (ii), we think of financial markets as a global coordination game (Manzano and Vives, 2011).

The stock market does not interact once and then clears for good. The New York Stock Exchange is open five days a week, as are nearly all stock exchanges. Today's trading therefore does not take place in informational isolation of the past or without concern for the future. An investor with a lowly opinion of some given firm may be faced with the situation where many other investors actually appear to believe in this firm and bought its stock. S/he may then come to believe that the company's prospects are actually better than first thought and buy its shares anyway. Such signaling, learning, and belief revision is an essential element of our sequential global games framework, suggesting the approach is well-suited to analyze and understand financial markets.

A somewhat similar reasoning applies to the housing market (Guren, 2018; Armona et al., 2019). The price of a house depends on the number of houses offered for sale as well as the demand for residential property. The price also depends on expectations of the (future) housing market (Kaplan et al., 2020) and the broader economy. When house prices are on the rise, individuals have been found to expect further rises in the future (Armona et al., 2019). Early buyers' and sellers' beliefs therefore influence the behavior of later buyers and sellers through the history of sales and prices. Moreover, it is not the case that all house owners at once decide to offer their houses for sale. Nor do all potential buyers wake up one morning and rush to their real estate agents simultaneously. And most individuals do not sell or buy a house repeatedly.

In sum, financial and real estate markets admit the key properties that make for a sequential global game. The theory developed in this paper may help us develop our understanding of them even further.

7 Discussion and Conclusions

This paper introduces sequential global games. We obtain several notable results. First, a sequential global game always has at least one equilibrium in monotone strategies. This is an existence result and does not exclude there may be more than one, or equilibria in other types of strategies. Second, when signals become sufficiently precise, the sequential global game has a unique equilibrium – there is only one strategy profile surviving iterated elimination of strictly dominated strategies, and it is in monotone strategies. Third, in expectations the unique equilibrium of a sequential global game is more efficient than that of the simultaneous move global game, everything else constant.

An interesting avenue for future research are sequential global games where the order of players is endogenous. In the present work, the player set was partitioned in an arbitrary and exogenous way. While there are no clear objections to such an approach from an abstract theory perspective, in applications it may matter. When cellphone were first introduced, there did not exist an exogenous black box spitting out who could buy the first generation of phones and who had to wait for the second. Rather, we may well imagine this was driven both by varying preferences among individuals and their differing beliefs regarding the usefulness of mobile phones. The latter case can be directly translated into global games language as differences in signals received at the start of the game.

The theoretical predictions made in simultaneous move global games are borne out in the laboratory, see in particular Heinemann et al. (2004, 2009). It would be interesting to see whether sequential global games are equally successful when put to test.

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A Proofs & Derivations

A.1 Proofs for Section 3

Proof of Proposition 1

Proof. By definition, β_0 and β_1 solve

$$g(\boldsymbol{t}_{\neg i}(\cdot; -\boldsymbol{\infty}), \beta_0) = 0, \tag{A.1}$$

and

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\infty}), \beta_1) = 0, \tag{A.2}$$

respectively. Define $\boldsymbol{\beta}_1 = (\beta_1, \beta_1, .., \beta_1)$ and $\boldsymbol{\beta}_0 = (\beta_0, \beta_0, .., \beta_0)$.

Any rational player i will neither play $s_i(b_i) = 1$ when $b_i < \beta_0$, nor $s_i(b_i) = 0$ when $b_i > \beta_1$, for such strategies are strictly dominated. The remaining, undominated strategies then satisfy the following inequalities: $t_i(b_i; \beta_0) \leq s_i(b_i) \leq t_i(b_i; \beta_1)$, for all $b_i \in \mathbb{R}$. Letting $S_i^1 \subseteq S_i$ denote the subset of these undominated strategies in S_i, one can conclude that player i will only play a strategy from $S_i^1 := \{s_i : \forall b_i \in \mathbb{R} : t_i(b_i; \beta_1) \leq s_i(b_i) \leq t_i(b_i; \beta_0)\}$, for all i. Out of completeness, define $S_{\neg i}^1 := \prod_{j \neq i} S_j^1$ and $S^1 := S_{\neg i}^1 \times S_i^1$, where a strategy-profile $s \notin S^1$ prescribes strictly dominated behavior to at least one player. Since strictly dominated strategies can be disregarded, players effectively play the *reduced game* which is the original game but with all strictly dominated strategies removed from the set of strategy profiles.

In the reduced game under S^1, too, g exhibits strategic complementarity. The highest expected gain to player i then realizes if all other players j play $s_j(b_j) = 1$ unless this is strictly dominated, which it is for all $b_j < \beta_0$. Hence, player i's highest gain obtains from the deleted strategy profile $\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_0)$, for see the definition of t in (**??**). Similarly, player i's lowest expected gain results if the other players play $t_{\neg i}(\cdot; \beta_{1,\neg i})$.[11] In the notation, for any b_i player i's highest gain, conditional on all other players not playing a strictly dominated strategy, is given by $g((t_{\neg i}(\cdot, \beta_0)), b_i)$, the lowest by $g((t_{\neg i}(\cdot, \beta_1)), b_i)$.

Now define β_0^1 as the point that solves:

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_0), \beta_0^1) = 0. \tag{A.3}$$

Similarly, define β_1^1 as the solution to:

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_1), \beta_1^1) = 0. \tag{A.4}$$

Note that $g(t_{\neg i}(\cdot; -\infty), \beta_0) = 0$ and $\beta_0 > -\infty$. From that, it follows that $g(\boldsymbol{t}_{\neg i}(\cdot; -\infty), b_i)$ $< g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_0), b_i)$ for any b_i. As β_0^1 solves (A.3), one concludes that $\beta_0^1 > \beta_0$. It can similarly be demonstrated that $\beta_1^1 < \beta_1$.

Hence, if player i knows that no player j will play a strictly dominated strategy, this means player i will expect a strictly negative gain from playing $x_i = 1$ for all $b_i < \beta_0^1$ or from playing $x_i = 0$ for all $b_i > \beta_1^1$. It follows that player i, knowing that no player j plays a strictly dominated strategy, will only play a strategy from $S_i^2 \subseteq S_i^1$, where $S_i^2 = \{s_i : \forall b_i \in \mathbb{R} : t_i(b_i; \beta_1^1) \leq s_i(b_i) \leq t_i(b_i; \beta_0^1)\}$.

One can repeat this procedure for any arbitrary number k of times, inductively defining β_0^{k+1} and β_1^{k+1} as the points that solve:

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_0^k), \beta_0^{k+1}) = g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_1^k), \beta_1^{k+1}) = 0. \tag{A.5}$$

If all players are rational and this rationality is common knowledge, no player i will therefore play a strategy not belonging to S_i^{k+1}, defined as:

$$S_i^{k+1} := \left\{ s_i : \forall b_i \in \mathbb{R} : t_i(b_i; \beta_1^k) \leq s_i(b_i) \leq t_i(b_i; \beta_0^k) \right\}. \tag{A.6}$$

[11] We mean to say that $t_{\neg i}(b_{\neg i}; \beta_0) = \sup_{s_{\neg i} \in S^1_{\neg i}} g(s_{\neg i}, b_i)$ and $t_{\neg i}(b_{\neg i}; \beta_1) = \inf_{s_{\neg i} \in S^1_{\neg i}} g(s_{\neg i}, b_i)$.

Which set of iteratively undominated strategies obtains if one repeats this process on and on? The following lemma will help answering that question.

Lemma 9. *For each player i in $\mathbb{P}$,*

(i) The action $x_i = 0$ is iteratively dominant at all $b_i < \beta_0^* \in (\beta_0, \beta_1)$, where β_0^* is the limit of the sequence $(\beta_0^k)_{k=0}^\infty$.

(ii) The action $x_i = 1$ is iteratively dominant at all $b_i > \beta_1^* \in (\beta_0, \beta_1)$, where β_1^* is the limit of the sequence $(\beta_1^k)_{k=0}^\infty$.

Proof. $g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{c}), b_i)$ is monotone increasing in b_i, and monotone decreasing in c. Moreover, $\beta_0 > -\infty$. Hence, if β_0 $(= \beta_0^0)$ solves $g(\boldsymbol{t}_{\neg i}(\cdot; -\boldsymbol{\infty}), \beta_0^0) = 0$ while β_0^1 solves $g(t_{\neg i}(\cdot; \boldsymbol{\beta}_0), \beta_0^1) = 0$, it must be that $\beta_0^0 < \beta_0^1$. By induction on this argument, it follows that $\beta_0^{k+1} > \beta_0^k$, for all k. Therefore, $(\beta_0^k)_{k=0}^\infty$ is a monotone increasing sequence. Any monotone sequence defined on a compact set (the interval $[\beta_0, \beta_1]$ is compact) converges to a point in the set. Hence, $(\beta_0^k)_{k=0}^\infty$ indeed has a limit and we label it β_0^*. Similarly, $(\beta_1^k)_{k=0}^\infty$ is a monotone (decreasing) sequence, which therefore has a limit, called β_1^*. *Q.E.D.*

Since $(\beta_0^k)_{k=0}^\infty$ and $(\beta_1^k)_{k=0}^\infty$ are converging, consecutive terms in either sequence become arbitrarily close to each other as $k \to \infty$.[12] Moreover, since conditional on $\boldsymbol{\beta}_0^k$ and $\boldsymbol{\beta}_1^k$, the points β_0^{k+1} and β_1^{k+1} are defined as the solution to (A.5), the limits β_0^* and β_1^* are characterized by:

$$g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_0^*), \beta_0^*) = g(\boldsymbol{t}_{\neg i}(\cdot; \boldsymbol{\beta}_1^*), \beta_1^*) = 0, \tag{A.7}$$

as given in the proposition. *Q.E.D.*

Proof of Lemma ??

Proof. By assumptions A2 and A3, $G_1(x_2; \beta)$ is increasing in x_2 and β. Hence, given b_1 and two strategies s_2 and s_2' where $s_2(h, b_2) \geq s_2'(h, b_2)$ for all h and all b_2, we have $\iint G_1(s_2(h, b_2), \beta)\, dF(\beta, b_2 \mid b_1) \geq \iint G_1(s_2'(h, b_2), \beta)\, dF(\beta, b_2 \mid b_1)$, where the inequality is strict if $s_2(h, b_2) > s_2'(h, b_2)$ for at least one b_2 and h. Hence, if we specify

[12] That is, for any real number $\nu > 0$, one can find a K_l such that $|\beta_l^{k+1} - \beta_l^k| < \nu$ for all $k \geq K_l$, for $l = 0, 1$.

the strategies s_2 and s_2' as follows:

$$s_2(h, b_2) = \begin{cases} t_2(b_2; \bar{b}^0) & \text{if} \quad h = 0 \\ t_2(b_2; \bar{b}^1) & \text{if} \quad h = 1 \end{cases},$$

and

$$s_2'(h, b_2) = \begin{cases} t_2(b_2; \bar{b}^0) & \text{if} \quad h = 0 \\ t_2(b_2; \bar{b}^{1'}) & \text{if} \quad h = 1 \end{cases},$$

we observe that $\iint G_1(s_2(h, b_2), \beta)\, dF(\beta, b_2 \mid b_1) > \iint G_1(s_2'(h, b_2), \beta)\, dF(\beta, b_2 \mid b_1)$ if and only of $\bar{b}^1 > \bar{b}^{1'}$. This proves part *(i)*. Part *(ii)* is proven in a symmetric way. *Q.E.D.*

Proof of Lemma 6

Proof. For given $\boldsymbol{b}_{2\setminus j}$ and b_j, part (i) is an immediate implication of the fact that j's posterior on $(\beta, \boldsymbol{b}_{2\setminus j})$ induced by $\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{b}}_1)$ lies to the right to the posterior on $(\beta, \boldsymbol{b}_{2\setminus j})$ induced by $\boldsymbol{t}_1(\cdot; \bar{\bar{\boldsymbol{b}}}_1)$, for any history h, if and only if $\bar{\boldsymbol{b}}_1 \geq \bar{\bar{\boldsymbol{b}}}_1$ (see Lemma 4 part (ii)). Since, ceteris paribus, G_j is increasing in both $\boldsymbol{x}_{2\setminus j}$ and β, this establishes part (i) of the lemma.

Given h, given the first-stage strategy profile $\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{b}}_1)$, and given b_j, player j's conditional posterior distribution on $(\beta, \boldsymbol{b}_{2\setminus j})$ is given. For any posterior on β, the probability that $\boldsymbol{b}_{2\setminus j} > \bar{\boldsymbol{b}}_{2\setminus j}$ is greater than the probability that $\boldsymbol{b}_{2\setminus j} > \bar{\bar{\boldsymbol{b}}}_{2\setminus j}$ if and only if $\bar{\bar{\boldsymbol{b}}}_{2\setminus j} > \bar{\boldsymbol{b}}_{2\setminus j}$. Since, given $h = \boldsymbol{x}_1$ and β, G_j is increasing in $\boldsymbol{x}_{2\setminus j}$, this establishes part (ii) of the lemma.

Given h and the first-stage strategy profile $\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{b}}_1)$, player j's conditional posterior distribution on $(\beta, \boldsymbol{b}_{2\setminus j})$ is first-order stochastically in increasing b_j (see Lemma 4 part (i)). Since, ceteris paribus, G_j is increasing in both $\boldsymbol{x}_{2\setminus j}$ and β, this establishes part (iii) of the lemma. *Q.E.D.*

A.2 Proofs for Section 4

A.3 Proofs for Section 5

Proof of Lemma 7

Lemma 10.

A.3.1 Forward and Backward Induction

Stage 1.

Suppose players in all stages $t > 1$ pursue the strategy $\boldsymbol{s}_t = \boldsymbol{t}(\cdot; \infty)$, for any possible history. Then we know that there exists a point $\beta_{1,1}$ that, for each player $i \in \mathbb{P}_1$, solves

$$g_1^{\sigma_\varepsilon}(\boldsymbol{t}_{1\setminus i}(\cdot; \boldsymbol{\infty}), \boldsymbol{t}_{s\setminus 1}(\cdot; \boldsymbol{\infty}), \beta_{1,1}) = 0. \tag{A.8}$$

Being true for every player $\mathbb{P}_1$, player i updates his beliefs and assumes that his fellow first-stagers play according to $\boldsymbol{t}_{1\setminus i}(\cdot; \boldsymbol{\beta}_{1,1})$. In this way, he finds a point $\beta_{1,1}^1$ that solves $g_1^{\sigma_\varepsilon}(\boldsymbol{t}_{1\setminus i}(\cdot; \boldsymbol{\beta}_{1,1}), \boldsymbol{t}_{s\setminus 1}(\cdot; \boldsymbol{\infty}), \beta_{1,1}^1) = 0$. Iteratively finding such points $\beta_{1,1}^k$, we obtain a sequence which, since $\beta_{1,1}^k \in [\beta_0, \beta_1]$ and $\beta_{1,1}^{k+1} < \beta_{1,1}^k$ for all k, converges to a limit. Call the limit $\bar{\beta}_{1,1}^0$.[13] By construction, the point $\bar{\beta}_{1,1}^0$ solves:

$$g_1^{\sigma_\varepsilon}(\boldsymbol{t}_{1\setminus i}(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0), \boldsymbol{t}_{s>1}(\cdot; \boldsymbol{\infty}), \bar{\beta}_{1,1}^0) = 0. \tag{A.9}$$

In words, it is a dominant strategy for each player i in stage 1 to play $x_i = 1$ whenever $b_i > \bar{\beta}_{1,1}^0$. Any strategy profile $\boldsymbol{s}_1 \leq \boldsymbol{t}_1(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0)$ is strictly dominated.

Stage 2.

Players in stage 2 know that $\boldsymbol{s}_1 \geq \boldsymbol{t}_1(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0)$. For a given history h_2, the *lowest* expected gain $g_2^{\sigma_\varepsilon}$, keeping strategies in stages $t > 2$ fixed, is therefore realized by assuming $\boldsymbol{s}_1 = \boldsymbol{t}_1(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0)$.

$$g_2^{\sigma_\varepsilon}(\boldsymbol{x}_1(\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0)), \boldsymbol{t}_{2\setminus j}(\cdot; \boldsymbol{\infty}), \boldsymbol{t}_{s>2}(\cdot; \boldsymbol{\infty}), \beta_{2,1}^{h_2,0}) = 0. \tag{A.10}$$

Because this is true for every player j in $\mathbb{P}_2$, we proceed as in the first stage a find a point $\bar{\beta}_{2,1}^{h_2,0}$, for every h_2, such that

$$g_2^{\sigma_\varepsilon}(\boldsymbol{x}_1(\boldsymbol{t}_1(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0)), \boldsymbol{t}_{2\setminus j}(\cdot; \bar{\boldsymbol{\beta}}_{2,1}^{h_2,0}), \boldsymbol{t}_{s>2}(\cdot; \boldsymbol{\infty}), \bar{\beta}_{2,1}^{h_2,0}) = 0. \tag{A.11}$$

Given that any strategy profile $\boldsymbol{s}_1 \leq \boldsymbol{t}_1(\cdot; \bar{\boldsymbol{\beta}}_{1,1}^0)$ is strictly dominated in the first stage, all strategy profiles $\boldsymbol{s}_2 \leq \boldsymbol{t}_2(\cdot; \bar{\boldsymbol{\beta}}_{2,1}^{h_2,0})$ are strictly dominated in the second stage, for any history h_2.

[13] The superscript zero may be somewhat flabbergasting. It it the superscript describing which found of iteration between stages we have reached.

Stage T.

$$g_T^{\sigma_\varepsilon}((\boldsymbol{x}_s(\boldsymbol{t}_s(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,0})))_{s<T}, \boldsymbol{t}_{T\setminus j}(\cdot;\boldsymbol{\infty}), \beta_{T,1}^{h_T,0}) = 0. \tag{A.12}$$

$$g_T^{\sigma_\varepsilon}((\boldsymbol{x}_s(\boldsymbol{t}_s(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,0})))_{s<T}, \boldsymbol{t}_{T\setminus j}(\cdot;\bar{\boldsymbol{\beta}}_{T,1}^{h_T,0}), \bar{\beta}_{T,1}^{h_T,0}) = 0. \tag{A.13}$$

Stage $T-1$, revisited.
We have constructed the thresholds $(\bar{\boldsymbol{\beta}}_{1,1}^{0}, \bar{\boldsymbol{\beta}}_{2,1}^{h_2,0}, ..., \bar{\boldsymbol{\beta}}_{T,1}^{h_T,0})$ by means of forward induction. Once we arrive at stage T, however, additional strategies may have become dominated in all preceding stages. Hence, we need to go back. Anticipating the strategy $\boldsymbol{t}_T(\cdot;\bar{\boldsymbol{\beta}}_{T,1}^{h_T,0})$, player i in stage $\mathbb{P}_{T-1}$

$$g_{T-1}^{\sigma_\varepsilon}(\boldsymbol{x}_{s<T-1}(\boldsymbol{t}_s(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,0})), \boldsymbol{t}_{T-1\setminus i}(\cdot;\bar{\boldsymbol{\beta}}_{T-1,1}^{h_{T-1},0}), \boldsymbol{t}_T(\cdot;\bar{\boldsymbol{\beta}}_{1,T}^{h_T,0}), \bar{\beta}_{T-1,1}^{h_{T-1},0}) > 0, \tag{A.14}$$

which follows from the facts that the gain $G_{T-1}(\boldsymbol{x}_{\neg i}, \beta)$ is strictly increasing in $\boldsymbol{x}_{\neg i}$ and $\bar{\beta}_{T,1}^{h_T,0} < \infty$ for all h_T. Performing some more iterated dominance within stage $T-1$, keeping $\boldsymbol{t}_{s>1}(\cdot;\bar{\boldsymbol{\beta}}_{1,s}^{h_s,0})$ fixed, we find a point $\bar{\beta}_{T-1,1}^{1}$ that solves:

$$g_{T-1}^{\sigma_\varepsilon}(\boldsymbol{x}_{s<T-1}(\boldsymbol{t}_s(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,0})), \boldsymbol{t}_{T-1\setminus i}(\cdot;\bar{\boldsymbol{\beta}}_{T-1,1}^{h_{T-1},1}), \boldsymbol{t}_T(\cdot;\bar{\boldsymbol{\beta}}_{1,T}^{h_T,0}), \bar{\beta}_{T-1,1}^{h_{T-1},1}) = 0, \tag{A.15}$$

Because $\bar{\beta}_{T,1}^{h_T,0} < \infty$ for all h_T, we can conclude that $\bar{\beta}_{T-1,1}^{h_{T-1},1} < \bar{\beta}_{T-1,1}^{h_{T-1},0}$.

Stage 1, revisited.
Given $\bar{\beta}_{T,1}^{h_T,0}$ and the resulting $\bar{\beta}_{T-1,1}^{h_{T-1},1}$, players in descending order of stages $s < T-1$ fins a thresholds $\bar{\beta}_{s,1}^{h_s,1}$ such that, in given stage t,

$$g_t^{\sigma_\varepsilon}(\boldsymbol{x}_{s<t}(\boldsymbol{t}_{s<t}(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,0})), \boldsymbol{t}_{t\setminus i}(\cdot;\bar{\boldsymbol{\beta}}_{t\setminus i,1}^{h_t,1}), \boldsymbol{t}_{s>t}(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,1}), \bar{\beta}_{t,1}^{h_t,1}) = 0. \tag{A.16}$$

When $t = 1$ we are back in the first stage and find the point $\bar{\beta}_{1,1}^{1}$ as follows

$$g_1^{\sigma_\varepsilon}(\boldsymbol{t}_{1\setminus i}(\cdot;\bar{\boldsymbol{\beta}}_{1,1}^{1}), \boldsymbol{t}_{s>1}(\cdot;\bar{\boldsymbol{\beta}}_{s,1}^{h_s,1}).\bar{\beta}_{1,1}^{1}) = 0. \tag{A.17}$$

A.3.2 Convergence and Equilibrium

Lemma 11. *The sequences* $(\bar{\boldsymbol{\beta}}^k_{1,1}, \bar{\boldsymbol{\beta}}^{h_2,k}_{2,1}, ..., \bar{\boldsymbol{\beta}}^{h_T,k}_{T,1})$ *and* $(\bar{\boldsymbol{\beta}}^k_{1,0}, \bar{\boldsymbol{\beta}}^{h_2,k}_{2,0}, ..., \bar{\boldsymbol{\beta}}^{h_T,k}_{T,0})$ *converge. We denote their limits* $(\bar{\boldsymbol{\beta}}^*_{1,1}, \bar{\boldsymbol{\beta}}^{h_2*}_{2,1}, ..., \bar{\boldsymbol{\beta}}^{h_T*}_{T,1})$ *and* $(\bar{\boldsymbol{\beta}}^*_{1,0}, \bar{\boldsymbol{\beta}}^{h_2*}_{2,0}, ..., \bar{\boldsymbol{\beta}}^{h_T*}_{T,0})$.

From the very construction of these sequences, it follows that for any stage $t = 1, 2, ..., T$ and for any history h_t (and $h_1 = \emptyset$)

$$g^{\sigma_\varepsilon}_t(\boldsymbol{x}_{s<t}(\boldsymbol{t}_{s<t}(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,1})), \boldsymbol{t}_{t\backslash i}(\cdot; \bar{\boldsymbol{\beta}}^{h_t*}_{t\backslash i,1}), \boldsymbol{t}_{s>t}(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,1}), \bar{\beta}^{h_t*}_{t,1}) = 0, \tag{A.18}$$

and

$$g^{\sigma_\varepsilon}_t(\boldsymbol{x}_{s<t}(\boldsymbol{t}_{s<t}(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,0})), \boldsymbol{t}_{t\backslash i}(\cdot; \bar{\boldsymbol{\beta}}^{h_t*}_{t\backslash i,0}), \boldsymbol{t}_{s>t}(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,0}), \bar{\beta}^{h_t*}_{t,0}) = 0. \tag{A.19}$$

Pick any stage t and any player $i \in \mathbb{P}_t$. If (i) the history of play h_t at the start of stage t is the realization of the strategy profile $\boldsymbol{t}_{s<t}(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,1})$, (ii) all players $j \neq i$ in $\mathbb{P}_t$ play according to the strategy profile $\boldsymbol{t}_{t\backslash j}(\cdot; \bar{\boldsymbol{\beta}}^{h_t*}_{t,1})$, and (iii) all players in stages $s > t$ are playing the strategy profile $\boldsymbol{t}_{s>t}(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,1})$, then the unique best response to player i in $\mathbb{P}_t$ is to play the strategy $t(\cdot; \bar{\boldsymbol{\beta}}^{h_t*}_{t,1})$, for any history h_t.

Proposition 3. *The strategy profiles* $(\boldsymbol{t}_s(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,1}))^T_{s=1}$ *and* $(\boldsymbol{t}_s(\cdot; \bar{\boldsymbol{\beta}}^{h_s*}_{s,0}))^T_{s=1}$ *are perfect Bayesian equilibria of the game.*

A.3.3 Limiting conditional distribution of β

For all $s = 1, 2, ..., T$, we define $n_s := |\mathbb{P}_s|$, the history $h_s := (\boldsymbol{x}_t)^s_{t=1}$, and $h_{s,1} := \sum_{i\in\mathbb{P}_s} x_i$, i.e. $h_{s,1}$ is the number of times $x_i = 1$ is played by players in stage s.

$$L(\beta) = \prod_{s=1}^{t-1}\left[1 - \Phi\left(\frac{\bar{b}^{h_{s,1}}_s - \beta}{\sigma_\varepsilon}\right)\right]^{h_{s,1}} \cdot \prod_{s=1}^{t-1}\left[\Phi\left(\frac{\bar{b}^{h_{s,1}}_s - \beta}{\sigma_\varepsilon}\right)\right]^{n_s - h_{s,1}} \cdot \phi\left(\frac{b_i - \beta}{\sigma_\varepsilon}\right) \cdot \phi\left(\frac{\beta - \bar{\beta}}{\sigma_\beta}\right) \tag{A.20}$$

Taking the natural logarithm of $L(\beta)$, differentiating with respect to β, we find:

$$\frac{b_i - \beta}{\sigma^2_\varepsilon} - \frac{\beta - \bar{\beta}}{\sigma^2_\beta} + \sum_{s=1}^{t-1}\frac{h_{s,1}}{\sigma_\varepsilon} m(\xi_s) - \sum_{s=1}^{t-1}\frac{n_s - h_{s,1}}{\sigma_\varepsilon} m(-\xi_s), \tag{A.21}$$

where $\xi_s := (\bar{b}^{h_{s,1}}_s - \beta)/\sigma_\varepsilon$ and $m(y) = \phi(y)/(1 - \Phi(y))$. Let $\hat{\beta}$ denote the value of point that solves:

$$\frac{b_i - \hat{\beta}}{\sigma_\varepsilon^2} - \frac{\hat{\beta} - \bar{\beta}}{\sigma_\beta^2} + \sum_{s=1}^{t-1} \frac{h_{s,1}}{\sigma_\varepsilon} m(\hat{\xi}_s) - \sum_{s=1}^{t-1} \frac{n_s - h_{s,1}}{\sigma_\varepsilon} m(-\hat{\xi}_s) = 0, \tag{A.22}$$

where $\hat{\xi} := (\bar{b}_s^{h_{s,1}} - \hat{\beta})/\sigma_\varepsilon$. Note that (A.22) is (i) monotone decreasing in β, (ii) monotone increasing in b_i, (iii) monotone increasing in $h_{s,1}$, all $s \leq t-1$, and (iv) monotone increasing in $\bar{b}_s^{h_{s,1}}$.

Multiply equation (A.22) by σ_ε^2 and rewrite $m(\hat{\xi}_s)\sigma_\varepsilon = (m(\hat{\xi}_s)/\hat{\xi}_s)(\bar{b}_s^{h_{s,1}} - \hat{\beta})$. Note that $m(x)/x \to 1$ if $x \to +\infty$ and $m(x)/x \to 0$ if $x \to -\infty$

In the limit as $\sigma_\varepsilon \to 0$, we can solve for $\hat{\beta}$

$$\hat{\beta}_i = \frac{b_i + \sum_{s=1}^{t-1} b_s^{h_s^1} [\mathbb{1}_{b_i,s} h_{s,1} + (1 - \mathbb{1}_{b_i,s})(n_s - h_{s,1})]}{1 + \sum_{s=1}^{t-1} [\mathbb{1}_{b_i,s} h_{s,1} + (1 - \mathbb{1}_{b_i,s})(n_s - h_{s,1})]}, \tag{A.23}$$

where $\mathbb{1}_{b_i,s}$ is the indicator function that takes value 1 if $b_i \geq \bar{b}_s^{h_{s,1}}$ and 0 otherwise.

Lemma 12. *Let σ_ε be sufficiently small. For all $t \leq T$, for all $(\bar{b}_s^{h_s})_{s=1}^T$, for any $\Delta > 0$,*

$$\begin{aligned} &g_t^{\sigma_\varepsilon}(\boldsymbol{x}_{s<t}(\boldsymbol{t}_{s<t}(\cdot; \bar{\boldsymbol{b}}_s^{h_s} + \boldsymbol{\Delta})), \boldsymbol{t}_{t\setminus i}(\cdot; \bar{\boldsymbol{b}}_{t\setminus i}^{h_t} + \boldsymbol{\Delta}), \boldsymbol{t}_{s>t}(\cdot; \bar{\boldsymbol{b}}_s^{h_s} + \boldsymbol{\Delta}), \bar{b}_t^{h_t} + \Delta) > \\ &g_t^{\sigma_\varepsilon}(\boldsymbol{x}_{s<t}(\boldsymbol{t}_{s<t}(\cdot; \bar{\boldsymbol{b}}_s^{h_s})), \boldsymbol{t}_{t\setminus i}(\cdot; \bar{\boldsymbol{b}}_{t\setminus i}^{h_t}), \boldsymbol{t}_{s>t}(\cdot; \bar{\boldsymbol{b}}_s^{h_s}), \bar{b}_t^{h_t}). \end{aligned} \tag{A.24}$$

Note that for any $t \leq T$ and any h_t, it must by construction hold that $\beta_{t,1}^{h_t *} \geq \beta_{t,0}^{h_t *}$.

A.3.4 Unique equilibrium

Define $\Delta_1 := \beta_{1,1}^* - \beta_{0,1}^*$. By construction, it is true that $\Delta_1 \geq 0$. Now suppose that in equilibrium, $\Delta_1 > 0$. Then, invoking Lemma 12, there must be at least one stage s and one history h_s such that $\beta_{1,s}^{h_s *} > \beta_{0,s}^{h_s *} + \Delta_1$. Defining $\Delta_s^{h_s} := \beta_{1,s}^{h_s *} - \beta_{0,s}^{h_s *}$, it follows that $\Delta_s^{h_s} > \Delta_1$.

Now if $\Delta_s^{h_s} > \Delta_1 > 0$ is to be an equilibrium, then either (i) there is at least one stage $r < s$ and at least one history h_r for which $\beta_{1,r}^{h_r *} < \beta_{0,r}^{h_r *}$ or (ii) there is at least one stage $t > s$ and at least one history h_t for which $\beta_{1,t}^{h_t *} > \beta_{0,t}^{h_t^*} + \Delta_s^{h_s}$. Option (i) is directly in contradiction with the fact that $\beta_{1,r}^{h_r *} \geq \beta_{0,r}^{h_r *}$ for all r and all h_r. Hence, the only possibility for $\Delta_s^{h_s} > \Delta_1 > 0$ to be consistent with an equilibrium is that there exists a stage $t > s$ for which $\beta_{1,t}^{h_t *} > \beta_{0,t}^{h_t^*} + \Delta_s^{h_s}$, so that for $\Delta_t^{h_t} := \beta_{1,t}^{h_t *} - \beta_{0,t}^{h_t *}$ we obtain

$\Delta_t^{h_t} > \Delta_s^{h_s} > \Delta_1 > 0$.

Repeating the above argument, rejection option (i) each time to support our hypothesis that $\Delta_1 > 0$ can be an equilibrium, we end up concluding that there is at least one history h_T for which $\beta_{1,T}^{h_T *} > \beta_{0,T}^{h_T *} + \Delta_1$. Can this be an equilibrium for players in stage T for some history h_T?

Suppose it is. It will be sufficient to look at the extreme case where $\beta_{1,1}^* - \beta_{0,1}^* > 0$ and $\beta_{1,t}^{h_t *} - \beta_{0,t}^{h_t *} = 0$ for all $T = 2, 3, ..., T-1$, all h_t. This most extreme case gives us the highest chance of finding that $\Delta_1 > 0$ might be an equilibrium since it yields the lowest posterior mean on β for players in stage T.

Note that a history h_T implies a list of sub-histories $h_2, h_3, ..., h_{T-1}$. Fix a particular list $\tilde{h}_2, \tilde{h}_3, ..., \tilde{h}_T$. We know that, for this particular history, the strategy profile $(\boldsymbol{t}_t(\cdot; \boldsymbol{\beta}_{0,t}^{\tilde{h}_t}))_{t=1}^T$ is an equilibrium of the game. Define $\tilde{\Delta}_t^{\tilde{h}_t} := \beta_{1,t}^{\tilde{h}_1} - \beta_{0,t}^{\tilde{h}_1}$.

The trick to showing that $\Delta_1 > 0$ cannot be an equilibrium is to find a signal $\tilde{b}_{1,T}$ such that the posterior on β is the same as in the "low" equilibrium.