Network Subsidies: Theory and Experiments
THEORY PART

Roweno J.R.K. Heijmans[1] and Sigrid Suetens[2]

[1]Swedish University of Agricultural Sciences
[2]Tilburg University

September 14, 2022

Abstract

1 Theory

There are N players, $N > 1$. With a slight abuse of notation, we let N denote both the set and the number of players. The action set of player i, $A_i \subseteq [0, 1]$, can be any closed, countable union of closed intervals and points that contain 0 and 1; we write $A = \times_j A_j$ and $A_{-i} = A \setminus A_i$. An action vector $a \in A$ is a vector of actions a_i for every player i; $a_{-i} \in A_{-i}$ denotes the action vector of all players but i. For convenience of notation, we define $\mathbb{1} = (1, 1, ..., 1)$ as the action vector a in which all players play 1 and $\mathbb{0} = (0, 0, ..., 0)$ as the vector in which all players play 0. The action vectors $\mathbb{1}_{-i}$ and $\mathbb{0}_{-i}$ are defined analogously. When comparing two vectors $x = (x_i)$ and $y = (y_i)$, the notation $x = y$ indicates that $x_i = y_i$ for all i. When we write $x \geq y$, we mean that $x_i \geq y_i$ for each i while also $x \neq y$.

Each player is endowed with a payoff function $u : A \to \mathbb{R}$. If a player chooses action i while her opponents play a_{-i}, her payoff is $u(a_i, a_{-i})$. Let $g_i(a_i, a'_i, a_{-i})$ denote the difference in a player's payoff when switching from action a_i to action a'_i given the others play a_{-i}; that is, $g_i(a_i, a'_i, a_{-i}) = u(a_i, a_{-i}) - u(a'_i, a_{-i})$. Naturally, $g_i(a_i, a'_i, a_{-i}) = -g_i(a'_i, a_i, a_{-i})$ and $g_i(a_i, a_i, a_{-i}) = 0$. The case of heterogeneous payoff functions is discussed in Section 1.5.

An action a_i is strictly dominant if player i's payoff is strictly higher when she plays a_i, rather than some $a'_i \neq a_i$, irrespective of the actions pursued by her opponents: $u(a_i, a_{-i}) > u(a'_i, a_{-i})$ for all $a'_i \neq a_i$ and all a_{-i}. An action vector $a \in A$ is a Nash equilibrium if $u(a_i, a_{-i}) \geq u(a'_i, a_{-i})$ for all $a'_i \neq a_i$ and all i; if the inequality is strict for all i, the equilibrium is called strict.

We maintain the following assumptions:

A1. *Strategic complementarities.* If $a_i > a_i'$ and $a_{-i} \geq a_{-i}'$ then $g_i(a_i, a_i', a_{-i}) > g_i(a_i, a_i', a_{-i}')$, for all i.

A2. *Positive spillovers.* If $a_{-i} \geq a_{-i}'$ then $u(a_i, a_{-i}) > u(a_i, a_{-i}')$ for all a_i and all i.

Assumption A1 makes the decision problem a coordination game in the sense that the incentive of each individual player to increase her action is increasing in the actions played by all other players. Note that A1 is an assumption on players' marginal payoffs. Assumption A2, in contrast, is an assumption on absolute payoffs, saying that each player is better off when other players increase their actions: playing a higher action is a public good. Because we are interested in coordination failures, we add to A1 the additional restriction that $g_i(1, a_i, \mathbb{1}_{-i}) > 0$ for all $a_i \neq 1$ and $g_i(0, a_i', \mathbb{0}_{-i}) > 0$ for all $a_i' \neq 0$.[1] Note that our theoretical results remain true under a slightly weakened version of assumption A1 that includes weakest-link games (Harrison and Hirshleifer, 1989; Van Huyck et al., 1990); we stick with A1 to simplify notation.

In most applications, the externalities agents impose on one another are hardly symmetric. Note that, following the literature (Segal, 1999, 2003; Bernstein and Winter, 2012; Sakovics and Steiner, 2012), our model allows that players generate asymmetric externalities. Given a vector of actions a_{-i}, let $\pi : A_{-i} \to A_{-i}$ denote a permutation function such that $\pi(a_{-i})$ is a vector which preseveres the elements of a_{-i} but reshuffles their order. The model allows that $g_i(a_i, a_i', a_{-i}) \neq g_i(a_i, a_i', \pi(a_{-i}))$; that is, we can accomodate the situation in which some players generate stronger spillovers than others.[2]

Assumptions A1 and A2 suffice to make the game a coordination game with multiple Pareto ranked equilibria. Coordination failure may hence occur. To see this, note that for each i assumptions A1 and A2 jointly imply $u(0, \mathbb{0}_{-i}) < u(0, \mathbb{1}_{-i}) < u(1, \mathbb{1}_{-i})$, so each player is strictly better off in the outcome $\mathbb{1}$ than in the outcome $\mathbb{0}$, though both outcomes are equilibria. A welfare-maximizing social planner, and indeed each individual player, strictly prefers coordination on $\mathbb{1}$.[3]

Proposition 1. *Both $a = \mathbb{0}$ and $a = \mathbb{1}$ are strict Nash equilibria of the game. Payoffs are strictly higher when players coordinate on $\mathbb{1}$.*

The fact that there is an equilibrium which is preferred by everyone while coordination on that equilibrium is by no means guaranteed creates room for policymaking. In what follows, we shall be concerned with the design of subsidies that incentivize coordination on the Pareto dominant equilibrium. Our focus will be on finding subsidies that make playing 1 strictly dominant for each player.

[1]This additional restriction is meant to exclude strictly dominant strategies as these would make our analysis largely uninteresting.

[2]That is, if we imagine the player set N as a (directed) graph in which the players are nodes, we allow that different nodes have different eigenvector centrality; see Elliott and Golub (2019) for an economics microfoundation and interpretation of eigenvector centrality in terms of price equilibria.

[3]While A2 implies that $\mathbb{0}$ and $\mathbb{1}$ are Nash equilibria, there may be other (not necessarily symmetric) ones. Let a denote such an equilibrium. Since $a \neq \mathbb{1}$, by A3 we have $u(a_i, a_{-i}) < u(a_i, \mathbb{1}_{-i})$ for all i. Moreover, A2 gives $u(a_i, \mathbb{1}_{-i}) < u(1, \mathbb{1}_{-i})$, so each individual player also prefers $\mathbb{1}$ over a.

1.1 Simple subsidies

We first study a *simple* subsidy. A subsidy is simple if the amount of subsidy received by a player depends only on her own action. Most actual subsidies appear to work this way.

Suppose each player is offered a simple subsidy equal to $s(a_i)$. We say that this subsidy makes playing 1 strictly dominant if and only if $u(1, a_{-i}) + s(1) > u(a_i, a_{-i}) + s(a_i)$ for all $a_i \neq 1$ and all $a_{-i} \in A_{-i}$. That is, the payoff (including subsidies) player 1 realizes from playing 1 should strictly exceed her payoff from playing any other action, irrespective of the actions chosen by her opponents. Since the planner wants to stimulate playing 1, we normalize s such that $s(a_i) = 0$ for all $a_i \neq 1$; the planner only subsidizes players who play 1. Subject to this normalization, the requirement that $s(a)$ makes playing 1 strictly dominant can written concisely as:

$$g_i(1, a_i, a_{-i}) + s(1) > 0, \tag{1}$$

for all $a_{-i} \in A_{-i}$ and all $a_i \neq 1$. Suppose then that the planner offers each player a subsidy $s(a_i)$ equal to:

$$s(a_i) = \begin{cases} g_i(0, 1, \mathbb{0}_{-i}) & \text{if} \quad a_i = 1, \\ 0 & \text{if} \quad a_i \neq 1. \end{cases} \tag{2}$$

Our next proposition establishes that the subsidy $s(a_i)$ given by (2) satisfies (1) and thus implements coordination on $\mathbb{1}$ in strictly dominant strategies.

Proposition 2. *Let the planner offer a subsidy equal to* (2)*. Then,*

(i) Playing $a_i = 1$ is a strictly dominant strategy for each player i;

(ii) Total subsidy spending is $\sum_{i=1}^{N} g_i(0, 1, \mathbb{0}_{-i}) > 0$ in equilibrium.

A potential objection to the simple subsidy (2) is that the amount of subsidy given to players is sufficient but not necessary to make playing 1 dominant. This would imply that equilibrium spending on subsidies reported in Proposition 2 is higher than necessary, generating an artificially costly image for the policy. It is easy to verify, however, that (2) is the least amount of fixed-rate subsidy required to make playing 1 strictly dominant.

Proposition 3. *Given the normalization that $s(a_i) = 0$ for all $a_i \neq 1$, so only playing 1 is subsidized, the minimum amount of subsidy required to make 1 strictly dominant is $s(1) = g_i(0, 1, \mathbb{0}_{-i})$.*

Rather that offer a subsidy on playing 1, the planner could levy a tax t on playing $a_i \neq 1$. Through the lens of a player's gain from playing 1 rather than some action $a_i \neq 1$, the effect of a tax on playing $a_i \neq 1$ is clearly equivalent to a subsidy (of the same size) on playing 1 of the same magnitude.[4] By the previous discussion, let the planner levy a tax $t(a_i)$ given by:

$$t(a_i) = \begin{cases} g_i(0, 1, \mathbb{0}_{-i}) & \text{if} \quad a_i \neq 1, \\ 0 & \text{if} \quad a_i = 1, \end{cases} \tag{3}$$

for all i. The following is an immediate corollary to Proposition 2.

[4]The statement is not that players' payoffs are unaffected by the swift from subsidies to taxes, which would be false. Their relative payoffs are unaffected, though.

Corollary 1. *Let the planner levy a tax equal to* (3). *Then,*

(i) Playing $a_i = 1$ is a strictly dominant strategy for each player i;

(ii) Total tax revenue is 0 in equilibrium.

1.2 Network subsidies

We next consider *network subsidies*, proposed by Heijmans (2021) for binary action coordination games. Like simple subsidies, a network subsidy entitles each player to a subsidy payment depending on her choice of action. In contrast to simple subsidies, however, the amount of network subsidy a player is entitled to also depends on the vector of actions a_{-i} chosen by her opponents. The intuition is that in coordination games, a player's incentive to increase her action depends on the actions pursued by all other players; hence, the (minimum) amount of subsidy required to make increasing her action to 1 dominant also depends on the actions of all others.

Let each player i be given a network subsidy equal to $s_i(a_i, a_{-i})$ when her action is a_i while her opponents play a_{-i}, for all $a_i \in A_i$ and all $a_{-i} \in A_{-i}$. We normalize the policy such that $s_i(a_i, a_{-i}) = 0$ for all $a_i \neq 1$. If player i plays 1, her net payoff (includig subsidies) is $u(1, a_{-i}) + s_i(1, a_{-i})$. If instead player i plays some other action $a'_i \neq 1$, her payoff is $u(a'_i, a_{-i})$. Her gain from choosing 1, rather than $a'_i \neq 1$, is therefore $g_i(1, a'_i, a_{-i}) + s_i(1, a_{-i})$. The requirement that a network subsidy s_i should make playing 1 strictly dominant means that

$$g_i(1, a'_i, a_{-i}) + s_i(1, a_{-i}) > 0, \tag{4}$$

has to be satisfied for all $a_{-i} \in A_{-i}$ and all $a'_i \neq 1$.

Condition (4) cannot be used directly as it depends on the alternative action a'_i the player was assumed to contemplate but which the planner does not observe. Recall then that $\mathbb{1}$ is a strict Nash equilibrium of the game, meaning that $g_i(1, a'_i, \mathbb{1}_{-i}) > 0$ for all $a'_i \neq 1$. This implies that the problem of finding a $s_i(1, a_{-i})$ that satisfies (4) can be relaxed to the more demanding, but easier to solve, approximation of the original problem of finding a $s_i(1, a_{-i})$ that solves $g_i(1, a'_i, a_{-i}) + s_i(1, a_{-i}) \geq g_i(1, a'_i, \mathbb{1}_{-i})$, or

$$s_i(1, a_{-i}) \geq g_i(1, a'_i, \mathbb{1}_{-i}) - g_i(1, a'_i, a_{-i}), \tag{5}$$

for all $a_{-i} \in A_{-i}$ and $a_i \neq 1$. Note that the right-hand side of (5) can be written as[5]

$$g_i(1, a'_i, \mathbb{1}_{-i}) - g_i(1, a'_i, a_{-i}) = g_i(1, 0, \mathbb{1}_{-i}) - g_i(a'_i, 0, \mathbb{1}_{-i}) - [g_i(1, 0, a_{-i}) - g_i(a'_i, 0, a_{-i})]. \tag{6}$$

We reshuffle the terms in (6), plug the outcome back in the right-hand side of (5), and get

$$s_i(1, a_{-i}) \geq g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}) - [g_i(a'_i, 0, \mathbb{1}_{-i}) - g_i(a'_i, 0, a_{-i})]. \tag{7}$$

Assumption (A1) implies that for any $a'_i \geq 0$ we have $g_i(a'_i, 0, \mathbb{1}_{-i}) - g_i(a'_i, 0, a_{-i}) \geq 0$, with a strict inequality for all $a'_i > 0$. It follows that

$$g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}) > g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}) - [g_i(a'_i, 0, \mathbb{1}_{-i}) - g_i(a'_i, 0, a_{-i})],$$

[5] One can always write the gain $g(a_i, a'_i, a_{-i})$ as the difference $g(a_i, a''_i, a_{-i}) - g(a''_i, a'_i, a_{-i})$ for any $a''_i \neq a_i, a'_i$. We choose $a''_i = 0$ so our analysis applies to any feasible action set A without further adjustments.

for all $a_{-i} \in A_{-i}$ and all $a_i' \in A_i \setminus \{0,1\}$. Using this, we see that a network subsidy s_i such that

$$s_i(1, a_{-i}) = g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}),$$

implies (7) and consequently (4), for all $a_i \neq 1$ and all $a_{-i} \in A_{-i}$.

To summarize, let the planner offer the following network subsidy scheme:

$$s_i(a_i, a_{-i}) = \begin{cases} g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}) & \text{if} \quad a_i = 1, \\ 0 & \text{if} \quad a_i \neq 1. \end{cases} \tag{8}$$

The network subsidy (8) has two key properties. First, like a simple subsidy, the amount of subsidy a player receives depends on her choice of action; in this case, the player receives a subsidy if and only if she plays 1, the action the planner would like all players to play.

Second, the amount of subsidy a player is entitled to depends on the actions chosen by her opponents. The intuition is that in a coordination game, playing 1 gets more attractive for each individual player as the number of other players who play 1 increases. The ex post amount of subsidy a player would need to strictly prefer playing 1, rather than any other action, is hence decreasing in the realized proportion of the population that plays 1. The network subsidy in (8) generally reflects this idea as it is decreasing in a_{-i}; for any two a_{-i} and a_{-i}' we have $s_i(1, a_{-i}) < s_i(1, a_{-i}')$ iff $a_{-i}' \geq a_{-i}$.

The dependence of a player's individual subsidy payment on the entire action vector a has an important implication: for all a_{-i} such that player i is willing to play 1 even without subsidies, the planner can offer a subsidy of zero without harming her incentive to play 1. We know there is at least one such vector as $\mathbb{1}$ is a strict Nash equilibrium of the game. Hence, a core property of the policy $s_i(a)$ is that players receive no subsidy payment when coordination on $\mathbb{1}$ is achieved: $s_i(1, \mathbb{1}_{-i}) = g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, \mathbb{1}_{-i}) = 0$. Moreover, recall that $s_i(a)$, by construction, makes playing 1 strictly dominant for each player i. It follows that coordination on $\mathbb{1}$ is the unique Nash equilibrium of the game with network subsidies and, therefore, that network subsidy spending is zero in equilibrium.

Proposition 4. *Let the planner offer a network subsidy equal to* (8)*. Then,*

(i) Playing 1 *is strictly dominant for each player i;*

(ii) Total subsidy spending is zero in equilibrium.

The network subsidy schemes $s_i(a)$ clearly yields the lowest equilibrium payment of all subsidies that induce efficient coordination in strictly dominant strategies – unless we allow taxing $a_i = 1$ (see Section 1.4 for a brief discussion), a subsidy less than zero is impossible. However, network subsidies are not necessarily the cheapest subsidy policy overall; the reason is that we solved for $s_i(a)$ using a series of sufficient, but not always necessary, conditions for strict dominance.

As was the case for simple subsidies, the planner might also levy a "network tax" on playing an action $a_i \neq 1$ rather than offer a network subsidy to playing $a_i = 1$. Let t_i denote this network tax. Suppose that:

$$t_i(a_i, a_{-i}) = \begin{cases} g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}) & \text{if} \quad a_i \neq 1, \\ 0 & \text{if} \quad a_i = 1. \end{cases} \tag{9}$$

The following proposition is an immediate corrollary to Proposition 4.

Corollary 2. *Let the planner levy a network tax equal to* (9)*. Then,*

(i) Playing 1 *is strictly dominant for each player i;*

(ii) Total tax revenues are zero in equilibrium.

Several papers on contracting in coordination games, including Sakovics and Steiner (2012) and Halac et al. (2020), consider threshold coordination games. In these games, agents invest into a common project which succeeds if and only if some critical level of investment is achieved. Define $\alpha := \sum_{i \in N} \omega_i a_i / N$, where ω_i are weights that satisfy $\omega_i \geq 0$ and $\sum_{i \in N} \omega_i = 1$. Take $A_i = \{0, 1\}$; we interpret $a_i = 1$ as investment, and $a_i = 0$ as no investment, by agent i. Let there be some threshold $I \in [0, 1]$ such that the payoff to an agent i who invests is $\overline{u}$ if $\alpha \geq I$ and $\underline{u}$ is $\alpha < I$; the payoff to not investing is normalized to 0. Assume that $\overline{u} > 0 > \underline{u}$. In this game, a network subsidy $s_i(a)$ on investing such that $s_i(a) = \overline{u} - \underline{u}$ if $\alpha < I$ and $s_i(a) = 0$ otherwise makes investment strictly dominant for each player i. For threshold coordination games, network subsidies thus satisfy the requirement of Halac et al. (2020) that contracts should be essentially bilateral, in that the planner offers each agent a subsidy payment "that does not depend on other agents' investment decisions except insofar as these decisions affect whether the project gets implemented (Halac et al., 2020)."

1.3 Off-equilibrium play and self-financed policy

Spending on network subsidies is zero only in equilibrium. Off-equilibrium behavior in which some players do, while others do not, play 1 leads to strictly positive spending on network subsidies. Although such a configuration of actions would not be expected from rational players in this policy environment, a planner might nonetheless be uncomfortable with the possbility. Here, we develop a self-financed network tax-subsidy policy that remedies those conderns.

Suppose the planner offers a network tax-subsidy scheme denoted $(\hat{s}_i, \hat{t}_i)$, where $\hat{s}_i$ and $\hat{t}_i$ are functions of the action vector a. $\hat{s}_i(a)$ represents the amount of subsidy paid to player i who chooses $a_i = 1$ in a; $\hat{t}_i(a)$ instead is the amount of tax levied on each player i who chooses $a_i \neq 1$ in a. Let $n(a)$ denote the number of players who choose 1 in a; formally, $n(a) = \sum_{i=1}^{N} I(a_i)$ where $I(a_i)$ is the indicator function that takes value 1 if $a_i = 1$ and 0 otherwise.

Let us hypothesize functions $\hat{s}$ and $\hat{t}$ such that, depending on the action vector a played, the network subsidy $\hat{s}_i$ received by player i is equal to $\hat{s}(a)$ if she plays 1 in a and 0 otherwise:

$$\hat{s}_i(a) = \begin{cases} \hat{s}(a) & \text{if} \quad a_i = 1, \\ 0 & \text{if} \quad a_i \neq 1. \end{cases} \tag{10}$$

Similarly, the network tax $\hat{t}_i$ levied on player i is equal to 0 if she plays 1 and $\hat{t}(a)$ otherwise:

$$\hat{t}_i(a) = \begin{cases} \hat{t}(a) & \text{if} \quad a_i \neq 1, \\ 0 & \text{if} \quad a_i = 1. \end{cases} \tag{11}$$

Given (10) and (11), when a is played total spending on subsidies in the network tax-subsidy scheme $(\hat{s}_i, \hat{t}_i)$ is given by:

$$n(a) \cdot \hat{s}(a), \tag{12}$$

whereas tax revenues are:

$$(N - n(a)) \cdot \hat{t}(a). \tag{13}$$

For a given vector of actions a, we define *net spending* as the sum total of subsidy payments minus the sum total of tax revenues, or $n(a) \cdot \hat{s}(a) - (N - n(a)) \cdot \hat{t}(a)$. To say that the $(\hat{s}_i, \hat{t}_i)$ is always self-financed means that we require net spending to be zero for all possible a:

$$n(a) \cdot \hat{s}(a) = (N - n(a)) \cdot \hat{t}(a). \tag{14}$$

The network tax-subsidy scheme $(\hat{s}_i, \hat{t}_i)$ makes playing 1 strictly dominant for each player i if it satisfies:

$$u(1, a_{-i}) + \hat{s}(a) > u(a_i, a_{-i}) - \hat{t}(a), \tag{15}$$

for all $a_i \neq 1$ and $a_{-i} \in A_{-i}$. We can rewrite this as:

$$g_i(1, a_i, a_{-i}) + \hat{s}(a) + \hat{t}(a) > 0, \tag{16}$$

for all $a_i \neq 1$ and $a_{-i} \in A_{-i}$. Using Proposition 4, we can conclude that in order to satisfy (16), the network tax-subsidy scheme must satisfy:

$$\hat{s}(a) + \hat{t}(a) = g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i}), \tag{17}$$

for each player i, all $a_i \neq 1$, and all $a_{-i} \in A_{-i}$. Combining the requirement of budget-neutrality (14) with the strict dominance constraint (17), we obtain:

$$N \cdot \hat{t}(a) = n(a) \cdot (\hat{s}(a) + \hat{t}(a)) = n(a) \cdot (g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i})), \tag{18}$$

where the first equality is obtained by adding $n(a) \cdot \hat{t}(a)$ to both sides of the equality in (14); the second equality follows from plugging (17) in the resulting relation. Condition (18) gives:

$$\hat{t}(a) = \frac{n(a)}{N} \cdot (g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i})). \tag{19}$$

Having found $\hat{t}(a)$, (17) immediately gives $\hat{s}(a)$:

$$\hat{s}(a) = \frac{N - n(a)}{N} \cdot (g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, a_{-i})) \tag{20}$$

The following proposition is now true by construction.

Proposition 5. *Let the planner offer a network tax-subsidy scheme $(\hat{s}_i, \hat{t}_i)$ in which each player is offered a subsidy $\hat{s}_i$ equal to* (10) *and levied a tax $\hat{t}_i$ equal to* (11)*. Let $\hat{t}(a)$ and $\hat{s}(a)$ be given by* (19) *and* (20)*, respectively. Then,*

(i) Playing 1 *is strictly dominant for each player i;*

(ii) Net spending is always zero.

Whether players end up coordination on the unique equilibrium $\mathbb{1}$ or not, net government spending on subsidies is always zero when the planner used the network tax-subsidy scheme $(\hat{s}_i, \hat{t}_i)$. This is desirable for two distinct reasons. First, it guarantees the planner a budget neutral policy even in the face of unexpected events.

Second, ex post budget neutrality renders the policy more effective. To understand what we mean by this, suppose it is publicly known that Congress has allocated to the planner a maximum budget of $B > 0$ to be spent on subsidies (this is similar in spirit to the "weak policymaker" in Sakovics and Steiner (2012)). Suppose also that there exist action vectors $\hat{a} \neq \mathbb{1}$ such that $\sum_{i \in N} s_i(\hat{a}) > B$, where s_i is the network subsidy given by (8). That is, there are (off-equilibrium) configurations of players' actions such that the sum total of network subsidies pledged by the planner exceed her budget. This possibility could undermine the entire policy. A network subsidy should make playing 1 strictly dominant. Strict dominance requires that playing 1 is the unique best response against any action vevtor a_{-i}. If the planner cannot pay the entire network subsisdy $s_i(\hat{a})$, whatever she *can* pay may be insufficient to make playing 1 the unique best response of player i against $\hat{a}_{-i}$. It follows that 1 need no longer be a strictly dominant strategy for player i, which could break equilibrium uniqueness. Since the total cost of the network tax-subsidy scheme $(\hat{s}_i, \hat{t}_i)$ cannot, by construction, exceed B, this policy is more effective than a policy of network subsidies only.

1.4 Taxing the desired outcome

In the network tax-subsidy scheme $(\hat{s}_i, \hat{t}_i)$, the planner levies taxes on the action she does not want players to play. We can go further. Suppose, for now, that an individual player i knows with certainty that all other players will play 1. Since $\mathbb{1}$ is a strict Nash equilibrium of the game, player i strictly prefers playing 1 to any other action $a_i \neq 1$. In this particular case, the planner could hence tax playing 1 up to indifference between 1 and i's next-best response to $\mathbb{1}_{-i}$ while leaving 1 a strict best response for player i. Moreover, recall that a network subsidy makes playing 1 strictly dominant for all players, implying that in the presence of a network subsidy policy the action vector $\mathbb{1}_{-i}$ is player i's only rational expectation about the behavior of all other players. It follows that the planner can both offer a network subsidy on playing 1 and levy a tax on playing 1 while leaving $\mathbb{1}$ the dominant strategy equilibrium of the game – total spending on network subsidies will be zero while tax revenues will be strictly positive in equilibrium.

More precisely, let $\tilde{a}$ denote the action that to each player i yields the second-highest payoff against $\mathbb{1}_{-i}$: $\tilde{a} = \arg\max_{a_i < 1} u(a_i, \mathbb{1}_{-i})$. Her loss from playing $\tilde{a}$, rather than 1, against $\mathbb{1}_{-i}$ is given by $g_i(1, \tilde{a}, \mathbb{1}_{-i}) > 0$.

Proposition 6. *Let the planner offer a network subsisdy equal to* (8)*. Let the planner also tax playing 1 at a rate* $t \in (0, g_i(1, \tilde{a}, \mathbb{1}_{-i}))$*. Then, in equilibrium,*

(i) Players coordinate on $\mathbb{1}$*;*

(ii) Spending on network subsidies is zero;

(iii) Tax revenues are $N \cdot t$*.*

The idea of taxing a desired outcome was also discussed by Bernstein and Winter (2012), who study contract design in a coordination game of investments among heterogeneous players. In their setup, the planner picks a subset of players who get subsidized enough to invest even if no player in the complement invests. The remaining players can then be taxed because they can take for granted investments by those who are subsidized. A similar idea can be found in Sakovics and Steiner (2012).

1.5 Heterogeneous players

The foregoing analysis maintained the strong assumption that all players have the same payoff function u. We next consider the case of heterogeneous players.

Let each player i be of an observable type $t \in \Theta$, where $\Theta = \{1, 2, ..., T\}$, $T \leq n$, is the set of types. A player of type t has utility function $u^t : A \to \mathbb{R}$. We define $g_i^t(a_i, a_i', a_{-i}) := u^t(a_i, a_{-i}) - u^t(a_i', a_{-i})$. It is assumed that u^t and g_i^t satisy assumptions (A1) and (A2) for all $t \in \Theta$ and that $\mathbb{0}$ and $\mathbb{1}$ continue to be strict Nash equilibria. To an individual i of type t, let the policymaker offer a network subsidy $s_i^t(a)$ equal to:

$$s_i^t(a_i, a_{-i}) \begin{cases} g_i^t(1, 0, \mathbb{1}_{-i}) - g_i^t(1, 0, a_{-i}) & \text{if} \quad a_i = 1, \\ 0 & \text{if} \quad a_i \neq 1, \end{cases} \tag{21}$$

for all i and t. The same argument used to establish that s_i makes playing 1 strictly dominant in the case of symmetric players will also reveal that s_i^t makes 1 strictly dominant when players are heterogeneous. We hence obtain the following generalization of Proposition 4.

Proposition 7. *Let the planner offer a type-secific network subsidy equal to* (21). *Then,*

(i) Playing 1 is strictly dominant for each player i;

(ii) Total subsidy spending is zero in equilibrium.

1.6 A simple example

Consider the symmetric 2×2 game in normal form given by Figure 1. It is straightforward to verify that this game satisfies assumptions A1-A2 and that it has two pure strategy strict Nash equilibria: $(0, 0)$ and $(1, 1)$. Of these, the strategy vector $(1, 1)$ is payoff-dominant while $(0, 0)$ is risk-dominant.

		Player 2	
		0	1
Player 1	0	$2, 2$	$3, 0$
	1	$0, 3$	$4, 4$

Figure 1: The first entry in each cell is the payoff to the row-player, the second that of the column-player.

We assume that players play the game once and choose their actions simultaneously; communication prior to the choice of actions is not allowed. Facing this coordination problem, it is not clear that players will settle on the Pareto dominant equilibrium or indeed on an equilibrium at all.

We first consider a simple subsidy. Plugging the payoffs from Figure 1 into (2), we obtain the following simple subsidy scheme:

$$s_i(a_i) = \begin{cases} 0 & \text{if} \quad a_i = 0, \\ 2 & \text{if} \quad a_i = 1, \end{cases} \tag{22}$$

for each i. To check that (22) works indeed, we note that $u(1,0)+s_i(1)-u(0,0) = 0+2-2 = 0$ and $u(1,1) + s_i(1) - u(0,1) = 4 + 2 - 3 = 3$. Conditional this simple subsidy, the payoff to playing 1 is hence guaranteed to be at least as high (and in some cases strictly higher) than the payoff to playing 0. Payoff-maximizing players will therefore choose 1. Since this implies that each player is entitled to receive the simple subsidy, total spending on subsidies is equal to $2 \times s_i(1) = 4$ in equilibrium.

From (8), the network subsidy scheme for this game is given by:

$$s_i(a_i, a_j) = \begin{cases} 0 & \text{if} \quad a_i = 0, \\ 2 & \text{if} \quad a_i = 1 \quad \text{and} \quad a_j = 0, \\ 0 & \text{if} \quad a_i = 1 \quad \text{and} \quad a_j = 1, \end{cases} \tag{23}$$

for each i. Observe that $u(1,0)+s_i(1,0)-u(0,0) = 0+2-2 = 0$ and $u(1,1)+s_i(1,1)-u(0,1) = 4 + 0 - 3 = 1$. Hence, conditional on the network subsidy given by (23) each player is better off playing 1, and the unique equilibrium of the game is $(1,1)$. Total spending on network subsidies is therefore $2 \times s_i(1,1) = 0$.

A Proofs

Proof of Proposition 2

Proof. (i). Playing 1 is strictly dominant if 1 gives i a strictly higher payoff than playing $a_i \neq 1$ irrespective of the actions played by all other players. Given the normalization that $s_i(a_i) = 0$ for all $a_i \neq 1$, the subsidy should hence satisfy $u(1, a_{-i}) + s_i(1) > u(a_i, a_{-i})$ for all $a_i \neq 1$, or $s_i(1) > g_i(a_i, 1, a_{-i})$ for all a_{-i}.

By assumption A1, for any $a'_{-i} \geq a_{-i}$ we have $g_i(1, a_i, a'_{-i}) \geq g_i(1, a_i, a'_{-i})$, with a strict inequality for all $a_i \neq 1$. Equivalently, $g_i(a_i, 1, a'_{-i}) \leq g_i(a_i, 1, a_{-i})$ with a strict inequality for all $a_i \neq 1$, so in particular $g_i(a_i, 1, a'_{-i}) \leq g_i(a_i, 1, \mathbb{0}_{-i})$. Moreover, as $\mathbb{0}$ is a strict Nash equilibrium (A2) we have $g_i(a_i, 1, \mathbb{0}_{-i}) \leq g_i(0, 1, \mathbb{0}_{-i})$ for all $a_i \neq 0$. Therefore, $g_i(a_i, 1, a'_{-i}) \leq g_i(a_i, 1, a'_{-i}) < g_i(0, 1, \mathbb{0}_{-i}) = s_i(1)$ for all $a_i \neq 0$ and all $a'_{-i} \neq \mathbb{0}_{-i}$. As we have shown that $s_i(1) > g_i(a_i, 1, a_{-i})$ is a sufficient condition for making 1 dominant, this proves the proposition.

(ii). Since (2) makes playing $a_i = 1$ strictly dominant for every player i, the unique equilibrium is $\mathbb{1}$. Therefore each player i receives $s_i(1) = g_i(0, 1, \mathbb{0}_{-i})$ in equilibrium. Summed over all players this gives $\sum_{i=1}^{N} g_i(0, 1, \mathbb{0}_{-i})$ in total subsidy spending. □

Proof of Proposition 3

Proof. We know from Proposition 2 that $s_i(1) = g_i(0, 1, \mathbb{0}_{-i})$ is generally enough, so it sufficies to find an action vector a_{-i} at which it is *just* enough. Therefore, consider $a_{-i} = \mathbb{0}_{-i}$. If $s_i(1)$ makes playing 1 dominant, then in particular it must solve:

$$u(1, \mathbb{0}_{-i}) + s_i(1) - u(a_i, \mathbb{0}_{-i}) \geq 0,$$

for all a_i. We can rewrite this condition as

$$s_i(1) \geq g_i(a_i, 1, \mathbb{0}_{-i}),$$

for all $a_i \neq 1$. This implies

$$s_i(1) \geq \max_{a_i} g_i(a_i, 1, \mathbb{0}_{-i}),$$

which by A2 gives

$$s_i(1) \geq g_i(0, 1, \mathbb{0}_{-i}).$$

□

Proof of Proposition 4

Proof. (i). By construction.

(ii). When s_i is offered, playing 1 is strictly dominant for each player i, see part (i) of the proposition. The unique Nash equilibrium of the game is therefore $\mathbb{1}_{-i}$. The equilibrium amount of network subsidy received by any player i is hence $s_i(a_i, a_{-i}) = s_i(1, \mathbb{1}_{-i}) = g_i(1, 0, \mathbb{1}_{-i}) - g_i(1, 0, \mathbb{1}_{-i}) = 0$. □

Proof of Proposition 6

Proof. In this policy environment, playing 1 is strictly dominant for player i iff:

$$g_i(1, 0, \mathbb{1}_{-i}) - t > g_i(1, 0, a_{-i}) - g_i(1, a_i, a_{-i}),$$

for all $a_i \neq 1$ and $a_{-i} \neq \mathbb{1}_{-i}$. We can rewrite the right-hand side,

$$g_i(1, 0, \mathbb{1}_{-i}) - t > g_i(a_i, 0, a_{-i}).$$

We know that $g_i(1, 0, \mathbb{1}_{-i}) > g_i(a_i, 0, \mathbb{1}_{-i}) \geq g_i(a_i, 0, a_{-i})$ for all $a_i \neq 1$ and $a_{-i} \neq \mathbb{1}_{-i}$. Hence, if t satisfies

$$t < g_i(1, a_i, \mathbb{1}_{-i}),$$

for all $a_i \neq 1$, playing 1 remains dominant. By definition, $\tilde{a} = \arg\max_{a_i \neq 1} g_i(a_i, 1, \mathbb{1}_{-i}) = \arg\min_{a_i \neq 1} g_i(1, a_i, \mathbb{1}_{-i})$. Hence, if $0 < t < g_i(1, \tilde{a}, \mathbb{1}_{-i})$, the planner can levy a a tax t on playing 1 while leaving 1 strictly dominant (provided a network subsidy is in place). Because 1 is strictly dominant for each player, $\mathbb{1}$ is the unique equilibrium of the game and spendong in network subsicies is zero. Total tax revenues are $N \cdot t > 0$ in equilibrium. □

References

Bernstein, S. and Winter, E. (2012). Contracting with heterogeneous externalities. *American Economic Journal: Microeconomics*, 4(2):50–76.

Elliott, M. and Golub, B. (2019). A network approach to public goods. *Journal of Political Economy*, 127(2):730–776.

Halac, M., Kremer, I., and Winter, E. (2020). Raising capital from heterogeneous investors. *American Economic Review*, 110(3):889–921.

Harrison, G. W. and Hirshleifer, J. (1989). An experimental evaluation of weakest link/best shot models of public goods. *Journal of Political Economy*, 97(1):201–225.

Heijmans, R. (2021). *On Environmental Externalities and Global Games*. PhD thesis, Tilburg University. CentER Dissertation Series Volume: 655.

Sakovics, J. and Steiner, J. (2012). Who matters in coordination problems? *American Economic Review*, 102(7):3439–61.

Segal, I. (1999). Contracting with externalities. *The Quarterly Journal of Economics*, 114(2):337–388.

Segal, I. (2003). Coordination and discrimination in contracting with externalities: Divide and conquer? *Journal of Economic Theory*, 113(2):147–181.

Van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *The American Economic Review*, 80(1):234–248.